

## Additional Math Formulae

### Summation Formulae

$$\sum_{n=k}^{N-1} \alpha^n = \begin{cases} N-k & \alpha = 1 \\ \frac{\alpha^k - \alpha^N}{1-\alpha} & \alpha \neq 1 \end{cases}$$

$$\sum_{n=k}^{\infty} \alpha^n = \frac{\alpha^k}{1-\alpha} \text{ if } |\alpha| < 1$$

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \text{ if } |\alpha| < 1$$

$$\sum_{n=0}^{\infty} n\alpha^n = \frac{\alpha}{(1-\alpha)^2} \text{ if } |\alpha| < 1$$

### Trigonometric Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta))$$

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha - \beta) + \sin(\alpha + \beta))$$

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

$$\sin(\theta) = \frac{1}{j2}(e^{j\theta} - e^{-j\theta})$$

### Series and Transform Equations

Discrete-Time Fourier Series $x[n] = \sum_{k=-N} X[k] e^{jk\Omega_0 n}$	$X[k] = \frac{1}{N} \sum_{n=-N} x[n] e^{-jk\Omega_0 n}$
Continuous-Time Fourier Series $x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$	$X[k] = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$
Discrete-Time Fourier Transform $x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$	$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$
Continuous-Time Fourier Transform $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$	$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$
Bilateral Laplace Transform $x(t) = \frac{1}{j2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$	$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$
Unilateral Laplace Transform $x(t) = \frac{1}{j2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$	$X(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt$
Bilateral $z$ -Transform $x[n] = \frac{1}{j2\pi} \oint X(z) z^{n-1} dz$	$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$
Unilateral $z$ -Transform $x[n] = \frac{1}{j2\pi} \oint X(z) z^{n-1} dz$	$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$

### Convolution

Convolution $x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t - \tau) d\tau$	$x[n] * y[n] = \sum_{v=-\infty}^{\infty} x[v] y[n-v]$
Periodic Convolution $x(t) \circledast y(t) = \int_T x(\tau) y(t - \tau) d\tau$	$x[n] \circledast y[n] = \sum_{v=-N}^N x[v] y[n-v]$
Circular Convolution $X(e^{j\Omega}) \circledast Y(e^{j\Omega}) = \int_{2\pi} X(e^{j\Gamma}) Y(e^{j(\Omega-\Gamma)}) d\Gamma$	

## Basic Continuous-Time Fourier Series Pairs

Name	Signal	Fourier Series
Basic Signal	$x(t)$ , Period $T$	$X[k]$ , $\omega_0 = \frac{2\pi}{T}$
Complex Exponential	$x(t) = e^{j p \omega_0 t}$	$X[k] = \delta[k - p]$
Cosine	$x(t) = \cos(p\omega_0 t)$	$X[k] = \frac{1}{2} (\delta[k - p] + \delta[k + p])$
Sine	$x(t) = \sin(p\omega_0 t)$	$X[k] = \frac{1}{j2} (\delta[k - p] - \delta[k + p])$
Constant	$x(t) = c$	$X[k] = c\delta[k]$
Periodic Square Wave	$x(t) = \begin{cases} 1, &  t  < T_1 \\ 0, & T_1 <  t  \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$X[k] = \frac{\sin(k\omega_0 T_1)}{k\pi}$
Impulse Train	$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$	$X[k] = \frac{1}{T}$

## Properties of Continuous-Time Fourier Series

Property	Periodic Signal	Fourier Series
Basic Signals	$x(t), y(t), z(t); T_x = T_y = T$	$X[k], Y[k], Z[k]; \omega_0 = \frac{2\pi}{T}$
Linearity	$z(t) = Ax(t) + By(t)$	$Z[k] = AX[k] + BY[k]$
Time Shifting	$z(t) = x(t - t_0)$	$Z[k] = X[k]e^{-jk\omega_0 t_0}$
Frequency Shifting	$z(t) = e^{jk_0 \omega_0 t} x(t)$	$Z[k] = X[k - k_0]$
Conjugation	$z(t) = x^*(t)$	$Z[k] = X^*[-k]$
Time Reversal	$z(t) = x(-t)$	$Z[k] = X[-k]$
Time Scaling	$z(t) = x(\alpha t), \alpha > 0$	$Z[k] = X[k], T_z = \frac{T_x}{\alpha}$
Periodic Convolution	$z(t) = \int_T x(\tau)y(t - \tau)d\tau$	$Z[k] = TX[k]Y[k]$
Multiplication	$z(t) = x(t)y(t)$	$Z[k] = \sum_{l=-\infty}^{\infty} X[l]Y[k-l]$
Differentiation	$z(t) = \frac{dx(t)}{dt}$	$Z[k] = jk\omega_x X[k]$
Integration	$z(t) = \int_{-\infty}^t x(\tau) d\tau, X[0] = 0$	$Z[k] = \left( \frac{1}{jk\omega_x} \right) X[k]$
Properties of Real Signals	$z(t)$ real	$\begin{cases} Z[k] = Z^*[-k] \\ \Re\{Z[k]\} = \Re\{Z[-k]\} \\ \Im\{Z[k]\} = -\Im\{Z[-k]\} \\  Z[k]  =  Z[-k]  \\ \angle Z[k] = -\angle Z[-k] \end{cases}$
Properties of Real, Even Signals	$z(t)$ real and even	$Z[k]$ real and even
Properties of Real, Odd Signals	$z(t)$ real and odd	$Z[k]$ imaginary and odd
Isolation of Even Part	$z(t) = x_e(t)$ with $x(t)$ real	$Z[k] = \Re\{X[k]\}$
Isolation of Odd Part	$z(t) = x_o(t)$ with $x(t)$ real	$Z[k] = j\Im\{X[k]\}$
Parseval's Relation (Power)	$P_{ave} = \frac{1}{T} \int_T  z(t) ^2 dt$	$P_{ave} = \sum_{k=-\infty}^{\infty}  Z[k] ^2$

## Basic Continuous-Time Fourier Transform Pairs

Name	Signal	Fourier Transform	Non-zero $X[k]$ (if periodic)
Basic Signal	$x(t)$	$X(j\omega)$	$X[k]$
Periodic Signal	$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$	$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$	$X[k]$
Complex Exponential	$x(t) = e^{j\omega_0 t}$	$X(j\omega) = 2\pi\delta(\omega - \omega_0)$	$X[1] = 1$
Cosine	$x(t) = \cos(\omega_0 t)$	$X(j\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$X[1] = \frac{1}{2}$ $X[-1] = \frac{1}{2}$
Sine	$x(t) = \sin(\omega_0 t)$	$X(j\omega) = \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$X[1] = \frac{1}{2j}$ $X[-1] = -\frac{1}{2j}$
Constant	$x(t) = c$	$X(j\omega) = 2\pi c\delta(\omega)$	$X[0] = c$
Periodic Square Wave	$x(t) = \begin{cases} 1, &  t  < T_1 \\ 0, & T_1 <  t  \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\sin(k\omega_0 T_1)}{k}\delta(\omega - k\omega_0)$	$X[k] = \frac{\sin(k\omega_0 T_1)}{k\pi}$
Impulse Train	$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$	$X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$X[k] = \frac{1}{T}$
Centered Rectangular Pulse	$x(t) = \begin{cases} 1, &  t  < T_1 \\ 0, &  t  > T_1 \end{cases}$	$X(j\omega) = \frac{2\sin(\omega T_1)}{\omega}$	not periodic
General Rectangular Pulse	$x(t) = u(t-a) - u(t-b)$ $a < b$	$X(j\omega) = \frac{2\sin(\omega(\frac{b-a}{2}))}{\omega} \exp\left(-j\omega\left(\frac{b+a}{2}\right)\right)$	not periodic
Sinc Function	$x(t) = \frac{\sin(Wt)}{\pi t}$	$X(j\omega) = \begin{cases} 1, &  \omega  < W \\ 0, &  \omega  > W \end{cases}$	not periodic
Impulse	$x(t) = \delta(t - t_0)$	$X(j\omega) = e^{-j\omega t_0}$	not periodic
Step	$x(t) = u(t - t_0)$	$X(j\omega) = \frac{e^{-j\omega t_0}}{j\omega} + \pi\delta(\omega)$	not periodic
Decaying Exponential	$x(t) = e^{-at}u(t)$ $\Re\{a\} > 0$	$X(j\omega) = \frac{1}{a + j\omega}$	not periodic
Decaying Polynomial Exp.	$x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$ $\Re\{a\} > 0$	$X(j\omega) = \frac{1}{(a + j\omega)^n}$	not periodic

## Properties of Continuous-Time Fourier Transforms

<b>Property</b>	<b>Signal</b>	<b>Fourier Transform</b>
Basic Signals	$x(t), y(t), z(t)$	$X(j\omega), Y(j\omega), Z(j\omega)$
Linearity	$z(t) = Ax(t) + By(t)$	$Z(j\omega) = AX(j\omega) + BY(j\omega)$
Time Shifting	$z(t) = x(t - t_0)$	$Z(j\omega) = e^{-j\omega t_0} X(j\omega)$
Frequency Shifting	$z(t) = e^{j\omega_0 t} x(t)$	$Z(j\omega) = X(j(\omega - \omega_0))$
Conjugation	$z(t) = x^*(t)$	$Z(j\omega) = X^*(-j\omega)$
Time and Frequency Scaling	$z(t) = x(at)$	$Z(j\omega) = \frac{1}{ a } X\left(\frac{j\omega}{a}\right)$
Convolution	$z(t) = x(t) * y(t)$	$Z(j\omega) = X(j\omega)Y(j\omega)$
Multiplication	$z(t) = x(t)y(t)$	$Z(j\omega) = \frac{1}{2\pi} X(j\omega) * Y(j\omega)$
Time Differentiation	$z(t) = \frac{d}{dt}x(t)$	$Z(j\omega) = j\omega X(j\omega)$
Integration	$z(t) = \int_{-\infty}^t x(\tau) d\tau$	$Z(j\omega) = \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$
Frequency Differentiation	$z(t) = tx(t)$	$Z(j\omega) = j\frac{d}{d\omega}X(j\omega)$
Properties of Real Signals	$z(t)$ real	$\begin{cases} Z(j\omega) = Z^*(-j\omega) \\ \Re\{Z(j\omega)\} = \Re\{Z(-j\omega)\} \\ \Im\{Z(j\omega)\} = -\Im\{Z(-j\omega)\} \\  Z(j\omega)  =  Z(-j\omega)  \\ \angle Z(j\omega) = -\angle Z(-j\omega) \end{cases}$
Properties of Real, Even Signals	$z(t)$ real and even	$Z(j\omega)$ real and even
Properties of Real, Odd Signals	$z(t)$ real and odd	$Z(j\omega)$ imaginary and odd
Isolation of Even Part	$z(t) = x_e(t)$ with $x(t)$ real	$Z(j\omega) = \Re\{X(j\omega)\}$
Isolation of Odd Part	$z(t) = x_o(t)$ with $x(t)$ real	$Z(j\omega) = j\Im\{X(j\omega)\}$
Parseval's Relation for Aperiodic Signals (Energy)	$E_{tot} = \int_{-\infty}^{\infty}  z(t) ^2 dt$	$E_{tot} = \frac{1}{2\pi} \int_{-\infty}^{\infty}  Z(j\omega) ^2 d\omega$

## Basic Bilateral Laplace Transform Pairs

Name	Signal	Laplace Transform	ROC
Basic Signal	$x(t)$	$X(s)$	$R_x$
Impulse	$x(t) = \delta(t - t_0)$	$X(s) = e^{-st_0}$	All $s$
Unit step	$x(t) = u(t - t_0)$	$X(s) = \frac{e^{-st_0}}{s}$	$\sigma > 0$
Reversed step	$x(t) = -u(-(t - t_0))$	$X(s) = \frac{e^{-st_0}}{s}$	$\sigma < 0$
Polynomial	$x(t) = \frac{t^{n-1}}{(n-1)!}u(t)$	$X(s) = \frac{1}{s^n}$	$\sigma > 0$
Reversed Polynomial	$x(t) = -\frac{t^{n-1}}{(n-1)!}u(-t)$	$X(s) = \frac{1}{s^n}$	$\sigma < 0$
Exponential	$x(t) = e^{-\alpha t}u(t)$	$X(s) = \frac{1}{s + \alpha}$	$\sigma > -\Re\{\alpha\}$
Reversed Exponential	$x(t) = -e^{-\alpha t}u(-t)$	$X(s) = \frac{1}{s + \alpha}$	$\sigma < -\Re\{\alpha\}$
Polynomial Exponential	$x(t) = \frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$X(s) = \frac{1}{(s + \alpha)^n}$	$\sigma > -\Re\{\alpha\}$
Rev. Poly. Exp.	$x(t) = -\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$X(s) = \frac{1}{(s + \alpha)^n}$	$\sigma < -\Re\{\alpha\}$
Cosine	$x(t) = \cos(\omega_0 t)u(t)$	$X(s) = \frac{s}{s^2 + \omega_0^2}$	$\sigma > 0$
Sine	$x(t) = \sin(\omega_0 t)u(t)$	$X(s) = \frac{\omega_0}{s^2 + \omega_0^2}$	$\sigma > 0$
Exponential Cosine	$x(t) = e^{-\alpha t} \cos(\omega_0 t)u(t)$	$X(s) = \frac{s + \alpha}{(s + \alpha)^2 + \omega_0^2}$	$\sigma > -\Re\{\alpha\}$
Exponential Sine	$x(t) = e^{-\alpha t} \sin(\omega_0 t)u(t)$	$X(s) = \frac{\omega_0}{(s + \alpha)^2 + \omega_0^2}$	$\sigma > -\Re\{\alpha\}$

## Properties of the Bilateral Laplace Transform

<b>Property</b>	<b>Signal</b>	<b>Laplace Transform</b>	<b>ROC</b>
Basic Signals	$x(t), y(t), z(t)$	$X(s), Y(s), Z(s)$	$R_x, R_y, R_z$
Linearity	$z(t) = Ax(t) + By(t)$	$Z(s) = AX(s) + BY(s)$	At least $R_x \cap R_y$
Time Shifting	$z(t) = x(t - t_0)$	$Z(s) = e^{-st_0}X(s)$	$R_x$
$s$ -domain Shifting	$z(t) = e^{s_0 t}x(t)$	$Z(s) = X(s - s_0)$	$s$ for $s - s_0 \in R_x$
Conjugation	$z(t) = x^*(t)$	$Z(s) = X^*(s^*)$	$R_x$
Time and Frequency Scaling	$z(t) = x(at)$	$Z(s) = \frac{1}{ a }X\left(\frac{s}{a}\right)$	$s$ for $\frac{s}{a} \in R_x$
Convolution	$z(t) = x(t) * y(t)$	$Z(s) = X(s)Y(s)$	At least $R_x \cap R_y$
Time Differentiation	$z(t) = \frac{d}{dt}x(t)$	$Z(s) = sX(s)$	At least $R_x$
Integration	$z(t) = \int_{-\infty}^t x(\tau)d\tau$	$Z(s) = \frac{1}{s}X(s)$	At least $R_x \cap \{\sigma > 0\}$
Frequency Differentiation	$z(t) = -tx(t)$	$Z(s) = \frac{d}{ds}X(s)$	$R_x$

## Properties of the Unilateral Laplace Transform

<b>Property</b>	<b>Signal</b>	<b>Laplace Transform</b>
Basic Signals	$x(t), y(t), z(t)$ $x(t) = y(t) = 0, t < 0$	$\mathcal{X}(s), \mathcal{Y}(s), \mathcal{Z}(s)$
Linearity	$z(t) = Ax(t) + By(t)$	$\mathcal{Z}(s) = A\mathcal{X}(s) + B\mathcal{Y}(s)$
Time Shifting	$z(t) = x(t - t_0)$	$\mathcal{Z}(s) = e^{-st_0}\mathcal{X}(s)$ if $x(t - t_0)u(t) = x(t - t_0)u(t - t_0)$
<i>s</i> -domain Shifting	$z(t) = e^{s_0 t}x(t)$	$\mathcal{Z}(s) = \mathcal{X}(s - s_0)$
Time and Frequency Scaling	$z(t) = x(at), a > 0$	$\mathcal{Z}(s) = \frac{1}{a}\mathcal{X}\left(\frac{s}{a}\right)$
Conjugation	$z(t) = x^*(t)$	$\mathcal{Z}(s) = \mathcal{X}^*(s^*)$
Convolution	$z(t) = x(t) * y(t)$	$\mathcal{Z}(s) = \mathcal{X}(s)\mathcal{Y}(s)$
Time Differentiation	$z(t) = \frac{d}{dt}x(t)$	$\mathcal{Z}(s) = s\mathcal{X}(s) - x(0^-)$
Frequency Differentiation	$z(t) = -tx(t)$	$\mathcal{Z}(s) = \frac{d}{ds}\mathcal{X}(s)$
Time Integration	$z(t) = \int_{0^-}^t x(\tau)d\tau$	$\mathcal{Z}(s) = \frac{1}{s}\mathcal{X}(s)$