Duke University
Edmund T. Pratt, Jr. School of Engineering

EGR 53L Fall 2007

Test II

Rebecca A. Simmons
Michael R. Gustafson II

Name (please print)______________________________

In keeping with the Community Standard, I have neither provided nor received any assistance on this test. I understand if it is later determined that I gave or received assistance, I will be brought before the Undergraduate Judicial Board and, if found responsible for academic dishonesty or academic contempt, fail the class. I also understand that I am not allowed to speak to anyone except the instructor about any aspect of this test until the instructor announces it is allowed. I understand if it is later determined that I did speak to another person about the test before the instructor said it was allowed, I will be brought before the Undergraduate Judicial Board and, if found responsible for academic dishonesty or academic contempt, fail the class.

Signature:______________________________

Problem I: [10 pts.] Binary

Convert the following numbers either into binary or into decimal notation. Be sure to clearly show your work in doing so, as merely reporting the correct answer will receive little credit.

(1) 100101.10011_2 to decimal
(2) 393_{10} to binary

$$100101.10011_2 = (1)2^5 + (0)2^4 + (0)2^3 + (1)2^2 + (0)2^1 + (1)2^0 + (1)2^{-1} + (0)2^{-2} + (0)2^{-3} + (1)2^{-4} + (1)2^{-5}$$
$$100101.10011_2 = 32 + 4 + 1 + (1/2) + (1/16) + (1/32) = 37 + (19/32) = 37.59375$$

Division	Result	Remainder
393/2	196	1
196/2	98	0
98/2	49	0
49/2	24	1
24/2	12	0
12/2	6	0
6/2	3	0
3/2	1	1
1/2	0	1

$393_{10} = 110001001$

Name (please print):
Community Standard (print ACPUB ID):

Problem II: [15 pts.] Finding Roots

(1) Given some function and its derivative:

$$f(x) = x^3 - x - e^x \qquad f'(x) = 3x^2 - 1 - e^x$$

and assume an initial guess for a root x_1=1.5, determine the *next* three approximations for a root of $f(x)$ using the Newton-Raphson method (that is, determine x_2, x_3, and x_4). Keep at least four significant figures. Also indicate the final values for the x tolerance and the f tolerance.

k	$x(k)$	$f(x(k))$	$f'(x(k))$	$-f/f'$	$x(k+1)$
1	$+1.5000e+00$	$-2.6067e+00$	$+1.2683e+00$	$+2.0552e+00$	$+3.5552e+00$
2	$+3.5552e+00$	$+6.3858e+00$	$+1.9229e+00$	$-3.3209e+00$	$+2.3433e-01$
3	$+2.3433e-01$	$-1.4855e+00$	$-2.0993e+00$	$-7.0762e-01$	$-4.7329e-01$
4	$-4.7329e-01$	$-2.5568e-01$	N/R	N/R	N/R

final $f_{\text{tol}} = |f(x(4))| = 2.5568e-01$

final $x_{\text{tol}} = |x(4) - x(3)| = 7.0762e-01$

(2) Given some function and its derivative:

$$f(x) = x^3 - x - e^x \qquad f'(x) = 3x^2 - 1 - e^x$$

and assuming an initial bracket of 0 to 4 (meaning an initial guess $x_1 = 2$), determine the *next* three approximations for a root of $f(x)$ using bisection (that is, determine x_2, x_3, and x_4). Keep four significant figures. Also indicate the final value for the f tolerance.

k	$x_l(k)$	$f(x_l(k))$	sign	$x_m(k)$	$f(x_m(k))$	sign	$x_u(k)$	$f(x_u(k))$	sign
1	0.00	$-1.0000e+00$	$-$	2.00	$-1.3891e+00$	$-$	4.00	$5.4018e+00$	$+$
2	2.00	$-1.3891e+00$	$-$	3.00	$+3.9145e+00$	$+$	4.00	$5.4018e+00$	$+$
3	2.00	$-1.3891e+00$	$-$	2.50	$+9.4251e-01$	$+$	3.00	$3.9145e+00$	$+$
4	2.00	$-1.3891e+00$	$-$	2.25	$-3.4711e-01$	$-$	2.50	$9.4251e-01$	$+$

final $f_{\text{tol}} = |f(x_m(4))| = 3.4711e-01$

note: final $x_{\text{tol}} = \frac{1}{2}(x_u(4) - x_l(4)) = 0.25$

Name (please print):
Community Standard (print ACPUB ID):

Problem III: [15 pts.] Finding Roots II

(1) Given some function $g(x) = 9x^5 + 4x^3 - 8x^2$ write the MATLAB code you would use to find all roots - including the complex and imaginary ones - for $g(x)$. How many roots - real, imaginary, or otherwise - will you get?

```
roots([9 0 4 -8 0 0])
% 5th order polynomial, 5 roots
```

(2) Given some equation $p\ \cos(r) = s\ \sin(s\ r)$ (note that r is the angle in the cos term but s times r is the angle in the sin term), write the MATLAB code you would use to find the s value for which the equation is true if $p = 2$ and $r = \frac{\pi}{6}$ Assume an initial guess of $s = 1$.

```
f = @(p, r, s) p*cos(r)-s*sin(s*r)
fzero(@(sD) f(2, pi/6, sD), 1)
```

(3) Given some equation $p\ \cos(r) = s\ \sin(s\ r)$, write the MATLAB code you would use to generate and plot an array of p values for which the equation is true if $r = \frac{\pi}{3}$ and s=linspace(0, 4). Assume an initial guess of $p = 0$. Plot p as a function of s using black squares. You do not need to label or title this plot nor do you need to save it.

```
f = @(p, r, s) p*cos(r)-s*sin(s*r)
s = linspace(0, 4);
for k=1:length(s)
    p(k)=fzero(@(pD) f(pD, pi/3, s(k)), 0);
end
plot(s, p, 'ks')
```

Note - this problem was not set up particularly well on Dr. G's part, since you could just say

$$p = \frac{s\ \sin(s\ r)}{\cos(r)}$$

and have your code be:

```
s = linspace(0, 4);
p = s.*sin(s*pi/3) ./ cos(pi/3)
plot(s, p, 'ks')
```

Ooops...

Name (please print):
Community Standard (print ACPUB ID):

Problem IV: [20 pts.] Matrix and Vector Calculations[1]

Given:

```
x = [-2 6 3 4]';
y = [7 5 -8 0]';
z = [5 -3; 2 8]; % note the ;
```

determine the following quantities by hand **and** then write the MATLAB code you would use to calculate them.

(1) a=$\|x\|_1$

$$\text{norm(x,1)} = \|x\|_1 = \sum_{k=1}^{N} |x_k| = 2+6+3+4 = 15$$

(2) b=$\|z\|_1$

$$\text{norm(z,1)} = \|z\|_1 = \text{maximum 1-norm of each column} = \max(7, 11) = 11$$

(3) c=$\|x\|_\infty$

$$\text{norm(x,inf)} = \|x\|_\infty = \max(x) = 6$$

(4) d=$\|z\|_\infty$

$$\text{norm(z,inf)} = \|z\|_\infty = \text{maximum 1-norm of each row} = \max(8, 10) = 10$$

(5) e=$\bar{y}$

$$\text{mean(y)} = \bar{y} = \text{average of } y = (7+5-8+0)/4 = 1$$

(6) f=$(S_t)_y$

$$\text{sum((y-mean(y)).\^{}2)} = (S_t)_y = \sum_{k-1}^{N} (y_k - \bar{y})^2 = (7-1)^2 + (5-1)^2 + (-8-1)^2 + (0-1)^2 = 134$$

(7) g=det(z)

$$\det(z) = (5)(8) - (-3)(2) = 46$$

(8) h=condition number of z, based on Frobenius norm

$$\text{cond(z, 'fro')} = \|z\|_e * \|z^{-1}\|_e \qquad z^{-1} = \frac{\begin{bmatrix} 8 & 3 \\ -2 & 5 \end{bmatrix}}{46}$$

$$\|z\|_e = \sqrt{(5)^2 + (2)^2 + (-3)^2 + (8)^2} \approx 10.1 \qquad \|z^{-1}\|_e = \frac{\sqrt{(8)^2 + (-2)^2 + (3)^2 + (5)^2}}{46} \approx 0.220$$

$$\text{cond(z, 'fro')} \approx 10.1 * 0.220 = 2.22$$

[1] **Did you put BOTH the MATLAB code AND the numerical calculations?????** *Yes.*

Name (please print):
Community Standard (print ACPUB ID):

Problem V: [25 pts.] Linear Algebra

A pressure sensor is attached to a fill tube near the base (0 mL) of a graduated cylinder and two voltage measurements are taken across the sensor. When there is 500 mL of fluid in the cylinder, the sensor produces 3 mV while 1000 mL of fluid produces 4 mV. A linear fit of the data may thus be obtained by solving the equations:

$$500m + b = 3$$
$$1000m + b = 4$$

where m is measured in mV/mL and b is measured in mV.

(1) Clearly show the matrix system defined by these equations.

(2) Clearly solve for the coefficients m and b using linear algebra. You must show your work - merely producing the correct answer will receive little credit. Also, you must solve using the inverse of the matrix - back-substitution may be used to check your work, but will also receive little credit.

(3) Calculate the condition number of this system by hand using the ∞-norm.

(4) Assuming that the data used to obtain the original system was calculated using five significant figures, how many significant figures are there in your answers? Note: you may report fractional values of significant figures.

(5) Another linear calibration is performed by taking readings at 900 mL and 1000 mL. Will the coefficients found for this system have more or fewer significant figures? You must prove your answer.

(6) Write the code you would need to perform all the calculations required for the numerical and discussion portion of this problem (i.e. parts 2 through 5). Use sensible (and clear!) variable names. For part 2, you must explicitly create variables `m` and `b` that contain... the m and b values.

$$\begin{bmatrix} 500 & 1 \\ 1000 & 1 \end{bmatrix} \begin{Bmatrix} m \\ b \end{Bmatrix} = \begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$$

$$\begin{Bmatrix} m \\ b \end{Bmatrix} = \begin{bmatrix} 500 & 1 \\ 1000 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} 3 \\ 4 \end{Bmatrix} = \frac{1}{500-1000} \begin{bmatrix} 1 & -1 \\ -1000 & 500 \end{bmatrix} \begin{Bmatrix} 3 \\ 4 \end{Bmatrix} = \frac{-1}{500} \begin{Bmatrix} (1)(3)+(-1)(4) \\ (-1000)(3)+(500)(4) \end{Bmatrix} = \begin{Bmatrix} \frac{1}{500} \\ 2 \end{Bmatrix}$$

$$\|A\|_\infty = 1001 \qquad \|A^{-1}\|_\infty = 1500/500$$

$$\text{cond(A, inf)} = \|A\|_\infty \|A^{-1}\|_\infty = 3003$$

```
4) Since log10(3003) = 3.4, you will lose about 4 significant figures,
leaving just 1.
```

$$\|A_2\|_\infty = 1001 \qquad \|A_2^{-1}\|_\infty = 1900/100$$

$$\text{cond(A2, inf)} = \|A\|_\infty \|A^{-1}\|_\infty = 19019$$

```
5) Since log10(19019) = 4.3, these answers would have fewer significant figures.
```

```
A = [500, 1; 1000 1];   % part of part 1
y = [3; 4];             % part of part 1
M = A\y                 % part of part 2
m = M(1)                % part of part 2
b = M(2)                % part of part 2
cond(A, inf)            % part 3
log10(cond(A, inf))     % part 4
A2 = [900 1; 1000 1]    % part of part 5
cond(A2, inf)           % part of part 5
log10(cond(A2, inf))    % part of part 5
```

Name (please print):
Community Standard (print ACPUB ID):

Problem VI: [15 pts.] Models and Estimation

A researcher performs an experiment and obtains population data as a function of time. Unsure of what model might work best for the data, the researcher has a graduate student analyze two different models - one is a straight line and one is an exponential. Your job is to repeat the analysis and draw a conclusion about the suitability of each model.

(1) The two models being considered have best-fit coefficients as given below:

$$\hat{p}_1(t) = 16.3t - 12.6$$
$$\hat{p}_2(t) = 0.0302e^{1.97t}$$

Given that, determine the estimate values for the following data set (you should carry three significant figures):

t_i	p_i	$\hat{p}_1(t_i)$	$\hat{p}_2(t_i)$
0	2	-12.6	0.030
1	3	3.7	0.217
2	5	20.0	1.55
3	10	36.3	11.1
4	80	52.6	79.8

(2) Determine the appropriate S_r, S_t, and r^2 values to provide evidence for any conclusions you will draw about the models. Be sure to clearly indicate what variable you are calculating and on what set of data or estimates you are operating.

$$\bar{p} = \frac{1}{5}(2 + 3 + 5 + 10 + 80) = 20$$

$$S_t = \sum_{n=1}^{5}(p_n - \bar{p})^2 = (2-20)^2 + (3-20)^2 + (5-20)^2 + (10-20)^2 + (80-20)^2 = 4538$$

$$S_{r,1} = \sum_{n=1}^{5}(p_n - \hat{p}_{n,1})^2 = (2-(-12.6))^2 + (3-3.7)^2 + (5-20)^2 + (10-36.3)^2 + (80-52.6)^2 \approx 1881$$

$$S_{r,2} = \sum_{n=1}^{5}(p_n - \hat{p}_{n,2})^2 = (2-0.030)^2 + (3-0.217)^2 + (5-1.55)^2 + (10-11.1)^2 + (80-79.8)^2 \approx 24.78$$

$$r_1^2 = \frac{S_t - S_{r,1}}{S_t} \approx \frac{4538-1881}{4538} = 0.586 \qquad r_2^2 = \frac{S_t - S_{r,2}}{S_t} \approx \frac{4538-24.78}{4538} = 0.995$$

(3) State which of the two models you believe provides a better mathematical fit and why.
The second model, with an r^2 much closer to the ideal case of 1, is a better mathematical fit. As an aside, because this includes population data, an exponential model is more likely accurate - but more information would be required to back that up.

(4) Write the code you would need to perform all the calculations required for this problem (get estimates and calculate appropriate S_r, S_t, and r^2 values). Use sensible (and clear!) variable names.

```
% Original data
t = [0  1  2  3  4]';
p = [2  3  5 10 80]';

% First fit - first-order polynomial
phat1 = polyval([16.3 -12.6], t);
% Second fit - general nonlinear fit
phat2 = (0.0302)*exp(1.97*t)

% Calculate statistical measures of goodness of fit
St  = sum((p-mean(p)).^2)
Sr1 = sum((p-phat1).^2)
Sr2 = sum((p-phat2).^2)
r21 = (St - Sr1) / St
r22 = (St - Sr2) / St
```

Note - if you actually had to determine the estimates, you could use:

```
% First fit - first-order polynomial
C1 = polyfit(t, p, 1);
phat1 = polyval(C1, t);
% Second fit - general nonlinear fit
fSSR = @(a, t, p) sum((p - a(1)*exp(a(2)*t)).^2);
C2 = fminsearch(@(a) fSSR(a, t, p), [1 1])
phat2 = C2(1)*exp(C2(2)*t)
```

Note 2 - the by-hand calculations part of this problem took far longer than we had intended given the point value of the problem...

Seriously - For Problem IV, did you put BOTH the MATLAB code AND the numerical calculations????? **YES!!! Leave me alone!**