Lab 9:
Linear Algebra

9.1 Introduction

This lab is aimed at demonstrating MATLAB's ability to solve linear algebra problems. At the end of the assignment, you should be able to write code that sets up and solves linear algebra problems for both a single set of coefficients and solutions as well as an array of situations. You should also determine the condition of the linear system and understand how that relates to the accuracy of the results. For this report, be sure to use `format short e` in your scripts and report back as many significant figures as does MATLAB. Make sure to use proper axis labels and titles and include your NET ID in the title.

9.2 Resources

The additional resources required for this assignment include:

- Books: Chapra
- Pratt Pundit Pages: MATLAB:Plotting Surfaces

9.3 Getting Started

1. Log into one of the PCs in the lab using your NET ID. Be sure it is set to log on to acpub.
2. Start a browser and point it to `http://pundit.pratt.duke.edu/wiki/Lab:B209`. Check out the `Using the PCs to Run MATLAB Remotely` section for how to get connected and make sure the connection is working.
3. Once connected to a machine you believe will also display graphics, switch into your `EGR53` directory and create a `lab9` directory inside it:

```
cd EGR53
mkdir lab9
```

4. Switch to your `~/EGR53/lab9` directory:

```
cd lab9
```

5. Copy all relevant files from Dr. G's public `lab9` directory:

```
cp -i ~mrg/public/EGR53/lab9/* .
```

   Do not forget the space and the "." at the end.
6. Open MATLAB by typing `matlab &` at the prompt that appears in your terminal window. It will take MATLAB a few seconds to start up.

9.4 Problems

9.4.1 Based on Chapra Problem 8.3, p. 209

First, re-cast the problem as a matrix equation:

$$[A]\{x\} = \{b\}$$

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} ? \\ ? \\ ? \end{Bmatrix}$$

then write a MATLAB script to solve for the unknowns and compute both the transpose and inverse of the $[A]$ matrix. Also, calculate the condition number of the system using the 1-norm, the 2-norm, the Frobenius norm, and the ∞-norm. Typeset the matrix equation, the transpose, and the inverse in your lab report. Report the values of the condition numbers in some professional-looking way, and state what those numbers mean with respect to the accuracy of your answer relative to the accuracy of the coefficients in the matrix.

9.4.2 Based Chapra 8.5, pp. 209-210

For this problem, you will be solving for the steady-state chemical concentrations in five different chemical reactors. First, note that the *flow rate* (Q) is the volume per unit time passing through each of the pipes. Assuming metric units, that means that 5 m^3/s of fluid is passing from left to right in pipe 01 and entering reactor 1. The *concentration* (c) is the mass of chemical contained per unit volume of fluid. Again assuming metric units, that means that there are 10 kg of chemical in each cubic meter of fluid passing through pipe 01. The two quantities together mean pipe 01 is bringing (5 m^3/s)(10 kg/m^3)=(50 kg/s) of chemical into reactor 1. This latter quantity is the *chemical mass flow rate* into the reactor.

For steady state, there will be a constant level of fluid in each of the five reactors as well as a constant chemical concentration in each. For the fluid levels to remain constant, the amount of fluid flowing into each reactor must equal the amount of fluid leaving each reactor. This gives rise to five equations of fluid flow (positive values are assumed to be flowing into the reactor):

Reactor 1	$Q_{01} - Q_{12} - Q_{15} + Q_{31} = 0$
Reactor 2	$Q_{12} - Q_{23} - Q_{24} - Q_{25} = 0$
Reactor 3	$Q_{03} + Q_{23} - Q_{31} - Q_{34} = 0$
Reactor 4	$Q_{24} + Q_{34} - Q_{44} + Q_{54} = 0$
Reactor 5	$Q_{15} + Q_{25} - Q_{54} - Q_{55} = 0$

This system has five equations and twelve unknowns at present. This makes it an *underdetermined* system - somehow, seven more pieces of information must be present in order to solve for the different pipe flow. The extra information may be obtained as a result of sensors on different pipes to establish the flow rate through them or through different principles of fluid dynamics involving the relative sizes of the pipes, dimensions and relative locations of the reactors, etc. For this *specific* problem, we are actually given the twelve flow rates at steady state and thus do not need to solve for them.

The problem, then, is to determine the mass concentrations c_i in the five reactors given the fluid flow rates Q_{xy} and the *inlet* concentrations (c_{01} and c_{03}). The concentrations in the pipes are determined by whatever reactor is providing the fluid in the pipe - for example, pipes 12 and 15 (where the pipe number is determined by the subscript of the fluid flow rate) both have concentrations of c_1 since that is the reactor containing the fluid in pipes 12 and 15. The mass flow rate through pipe 12 is thus $Q_{12}c_1$. Similarly, the mass flow rate into the system through pipe 03 would be $Q_{03}c_{03}$ while the mass flow rate out of the system through pipe 44 would be $Q_{44}c_4$.

There are thus five unknowns (the five reactor concentrations) and five equations (the conservation of chemical mass in each of the reactors). You will write the five equations and then use MATLAB to solve for the five concentration levels. You will present your equations in matrix form as shown below. Note that the conservation of mass equation for the first reactor, which is:

$$\text{Mass out} = \text{Mass in}$$
$$Q_{12}c_1 + Q_{15}c_1 = Q_{01}c_{01} + Q_{31}c_3$$

should be re-written to put the unknowns on the left and the known values on the right as:

$$(Q_{12} + Q_{15})c_1 + (-Q_{31})c_3 = (Q_{01})c_{01}$$

The result is included in the typeset equation below as a reference. Note below that positive signs on the *left* indicate fluid flowing out of a particular reactor while positive signs on the *right* indicate fluid flowing in:

$$\begin{bmatrix} Q_{12}+Q_{15} & 0 & -Q_{31} & 0 & 0 \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{Bmatrix} = \begin{Bmatrix} Q_{01}c_{01} \\ ? \\ ? \\ ? \\ ? \end{Bmatrix}$$

Once you obtain this, you should calculate the four condition numbers of the system - you will report these in your lab as well as what they mean relative to the accuracy of the flow rate and inlet concentration measurements.

Next - in this same script - solve for all five unknown concentrations given the inlet concentrations in the book (i.e. $c_{01} = 10$ kg/m^3 and $c_{03} = 20$ kg/m^3). As a quick check on your program, you may want to test it using different values of the inlet concentrations for which the solution is easy. For example, if you use 0 kg/m^3 for both inlet concentrations you should get 0 kg/m^3 for *all* tank concentrations. If you put in the same value for each inlet concentration, you should get that same concentration in each of the tanks. Once you believe your program is working, reinstate the values specified in the book, solve for the tank concentrations, and present the values of all five tank concentrations in the lab report. As noted above, the flow rate (Q_{ij}) values are in m^3/s and the concentration (c_{0j} and c_i) values are in kg/m^3. This means the rate of mass flow measurements ($Q_{0j}c_{0j}$ or $Q_{ij}c_i$) in kg/s. For the most part, the values you report will be concentrations. This completes your first task - you should thus save the script at this point, then you may want to "save as" with a slightly different name to start another script to complete your next task.

From this point forward, you are going to focus only on the concentrations in Tanks 4 and 5 since those are the tanks connected to the output ports. Your linear algebra system will still solve for all five concentrations in a given situation, but you will only need to *store* and *present* the values for those two tanks. For your second task, you will expand your first program to solve for a variety inlet concentrations for Tank 3. That is, you will hold c_{01} fixed at 10 kg/m^3 as before, but will sweep c_{03} through 101 values between 0 and 20 kg/m^3. The reason for 101 values (versus, say, the usual 100) is to make sure that c_{03} is exactly 10 kg/m^3 at some point - that would be a good answer to check to make sure c_4 and c_5 are also 10 kg/m^3. You should also check to make sure that the *final* values calculated match those from your first script since the situations match.

To make the modifications to this second script, you need to consider which parts of the matrix equation you should change and how you will store the appropriate elements from $\{c\}$ in two vectors that will hold onto 101 different values each for c_4 and for c_5. Once you have performed these calculations, make two plots - one of c_4 as a function of c_{03} and the other of c_5 as a function of c_{03}. Be sure to include proper units on your graphs - you will be plotting one concentration as a function of another concentration. This completes your second task - you should save this second script and then "save as" to start a third (and final) script.

Your third task will be to look at 315 different combinations of c_{01} and c_{03} with the goal being to generate two surface plots, with contours, of c_4 and c_5 each as functions of c_{01} and c_{03}. Modify your programming structure to handle a situation where there is a range of c_{01} values and, for each one of those, a range of c_{03} values. Put some thought into how you want to store the information you are calculating. Remember - at the end of the day you will make two surface plots with contours so make sure your data is stored appropriately. Think twists.[1] And `meshgrid`.[2] And how happy you are Lab 9 only has two problems.[3]

For these graphs, assume c_{01} can take on 15 linearly spaced values between 0 and 20 kg/m^3 while c_{03} can independently take on 21 linearly spaced values between 0 and 20 kg/m^3. The reason for having different resolutions is to help you make sure you are building your matrix of values correctly - if you have something reversed, MATLAB will give you an "out of bounds" warning. When you make your surface plot, be sure there are more values for c_{03} than there are for c_{01}. Again, be sure to include proper units on your graphs - each dimension is in terms of a concentration.

For the lab report, you should have the following:

(1) The typeset version of the matrix equation using the *symbolic* representation of the flow rates and the concentrations.

(2) The calculations and meanings of the condition number of the system using the *specific* numerical values of the flow rates as given in the book. Since these flow rates will not change throughout the assignment, the condition numbers are constant as well.

(3) The results of the first script, which will report the values of the five tank concentrations given $c_{01} = 10$ kg/m^3 and $c_{03} = 20$ kg/m^3. Be sure to include proper units and organize the solution in some professional manner (i.e. a tabular of some kind). Note that this script does *not* produce a graph of any kind.

(4) The plots from the second script, which will show how the concentrations in Tanks 4 and 5 change given $c_{01} = 10$ kg/m^3 as the concentration c_{03} goes from 0 to 20 kg/m^3. There should be a total of 101 data points, and you should plot these using a black line. Be sure to include proper axis labels (with units) and titles.

(5) The surfaces with contours from the third script, which will show how the concentrations in Tanks 4 and 5 change given c_{01} and c_{03} can each range from 0 to 20 kg/m^3. Each surface should be comprised of 315 nodes, and the surface should be created using the `surfc` command. Be sure to save these figures using the `-depsc` tag to have the printers use grayscale. As always, be sure to include proper axis labels (with units) and titles.

(6) The scripts themselves.

(7) The following discussions: Based on all of the above, describe how you think the output concentrations depend on the inlet concentrations. Which inlet concentrations seem to have the largest impact on each output concentration? Which inlet concentration seems to have the largest impact overall?

[1] Palm 5.40, p. 350.
[2] Pratt Pundit, `MATLAB:Plotting Surfaces`
[3] *Very.*