Laboratory 5: Solving Circuits With Capacitors

5.1 Introduction

This lab focuses on determining the ordinary differential equations involved with circuits which include capacitors and on using Maple to both find and plot the solutions to those equations. Given that, be sure to look at the Pratt Pundit Pages `Maple/Plotting` and `Maple/Differential Equations`.

5.2 Solution Techniques with Capacitors

Although the current/voltage relationship for a capacitor is defined as a first order differential equation or by an integral:

$$i_{\text{C}}(t) = C\frac{dv_{\text{C}}(t)}{dt} \qquad\qquad v_{\text{C}}(t) = \frac{1}{C}\int_{-\infty}^{t} i_{\text{C}}(\tau)\ d\tau = v_{\text{C}}(t_0) + \frac{1}{C}\int_{t_0}^{t} i_{\text{C}}(\tau)\ d\tau$$

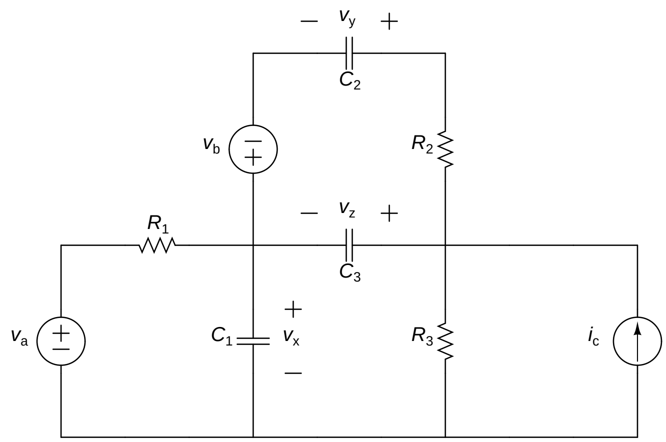
the methods for finding equations in a circuit - Node Voltage, Branch Current, and Mesh Current - remain valid. Because you will generally want to write differential equations without using integration, you will want to find currents in terms of voltages. For this reason, the Node Voltage Method - and its dependence on Kirchhoff's Current Law - may be best suited to generating the modeling equations for a circuit containing sources, resistors, and capacitors.

5.3 Initial Conditions

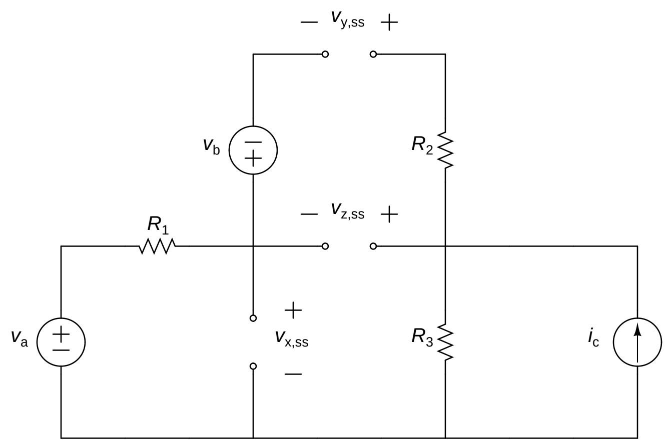
Because capacitors are capable of storing energy (recall that $E = \frac{1}{2}Cv^2$), they can have a "memory." This means solving circuits including capacitors will require knowing some initial state for the energy storage units. An analogy to this is filling a bucket from a tap - knowing the flow rate into the bucket is important information to have, but you also must know how much water was in the bucket at some point in order to determine the amount at other times. For capacitors, knowing the current through the capacitor is important information to have, but you also must know what the voltage was at some point in order to determine the amount at other times. One nice feature of capacitors is that the voltage cannot change instantaneously.

There are several different ways to determine the initial state of a circuit. By far the easiest is when the problem statement tells you a value. This can be done directly ("Assume the capacitor voltage is 6 V at time t=0") or slightly less directly ("Assume the circuit initially stores no energy" or "Assume the capacitor is storing 10 J").

Absent this information, you may have to analyze the circuit in some DC steady-state configuration to determine the initial capacitor voltage. For this, it is important to note that for most circuits with all constant sources, a capacitor will reach some steady state voltage and will then remain there indefinitely.[1] This means, for most circuits with constant sources, the long-term behavior of v_C is to remain constant - which in turn indicates a capacitor current $i_\mathrm{C} = 0$ since $i_\mathrm{C} = C\ \frac{dv_\mathrm{C}}{dt}$. Therefore, the long term behavior of the capacitor is equivalent to that of an *open circuit.* For example, given the following circuit with known voltage source values v_a and v_b, known current source value i_c, and known passive component values:

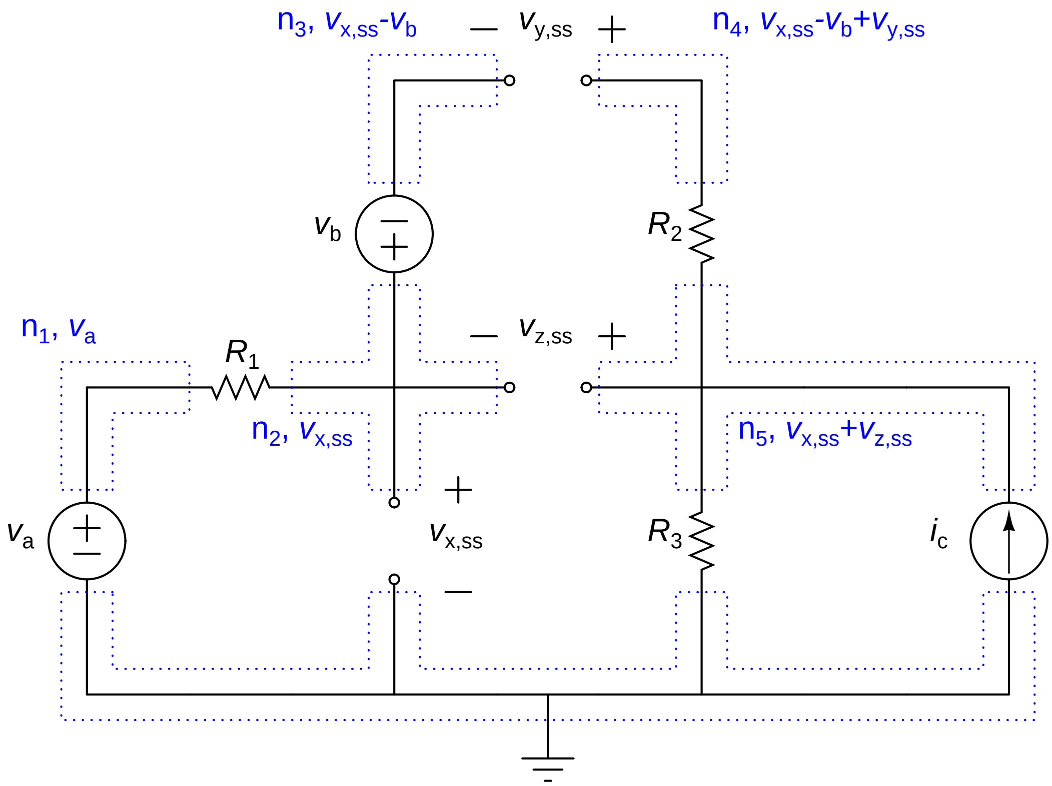


you may need to determine expressions for the voltages across the capacitors (labeled as v_x, v_y, and v_z). To get steady state values for these values, assuming constant sources, note that the current through each of the capacitors must be 0. An equivalent circuit drawing to determine the steady state values would therefore be:



Since the unknowns for which you are solving are all voltage values, it might be most efficient to use Node Voltage Method to determine the unknowns. There are several other techniques that could be used here - superposition along with voltage and current division, for example - but this handout will use Node Voltage Method. Even given that, there are different choices for how to label the nodes. For example, looking at the circuit you should see that the only branches in which the current can be non-zero would be in R_3 and i_c. You could therefore start by labeling the voltage drop across R_3 as $R_3 i_\mathrm{c}$. However, for this example, we will use known voltages, followed by labeled unknown voltages, followed by new unknowns. Given that, the circuit would thus be labeled as:

[1]There are some counterexamples to this: circuits with a current source in series with a capacitor, for example, or circuits where the capacitor does not "see" a positive resistance.



The useful KCL equations (avoiding voltage sources and using supernodes appropriately) are:

$$\text{KCL, sn23} \qquad \frac{(v_{\text{x,ss}} - v_{\text{a}})}{R_1} = 0$$

$$\text{KCL, n4} \qquad \frac{((v_{\text{x,ss}} - v_{\text{b}} + v_{\text{y,ss}}) - (v_{\text{x,ss}} + v_{\text{z,ss}}))}{R_2} = 0$$

$$\text{KCL, n5} \qquad \frac{((v_{\text{x,ss}} + v_{\text{z,ss}}) - (v_{\text{x,ss}} - v_{\text{b}} + v_{\text{y,ss}}))}{R_2} + \frac{((v_{\text{x,ss}} + v_{\text{z,ss}}))}{R_3} - i_{\text{c}} = 0$$

Note that solving these simultaneously yields:

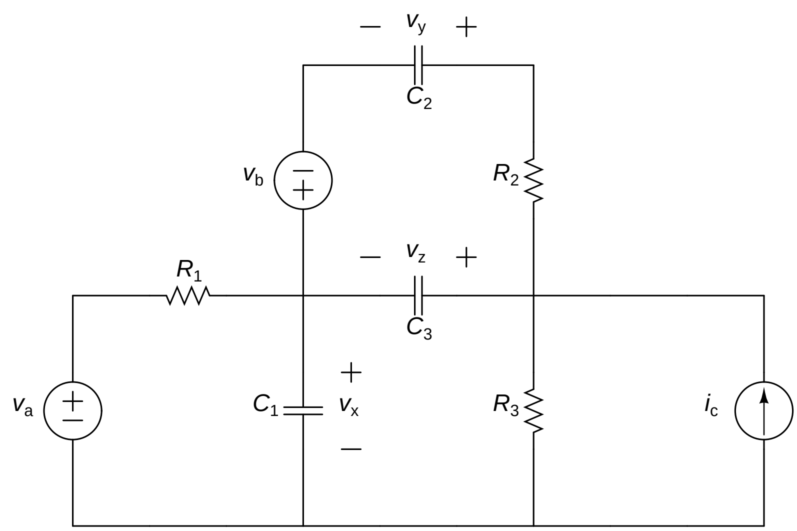
$$v_{\text{x,ss}} = v_{\text{a}}$$
$$v_{\text{y,ss}} = R_3 i_{\text{c}} - v_{\text{a}} + v_{\text{b}}$$
$$v_{\text{z,ss}} = R_3 i_{\text{c}} - v_{\text{a}}$$

5.4 Differential Equations

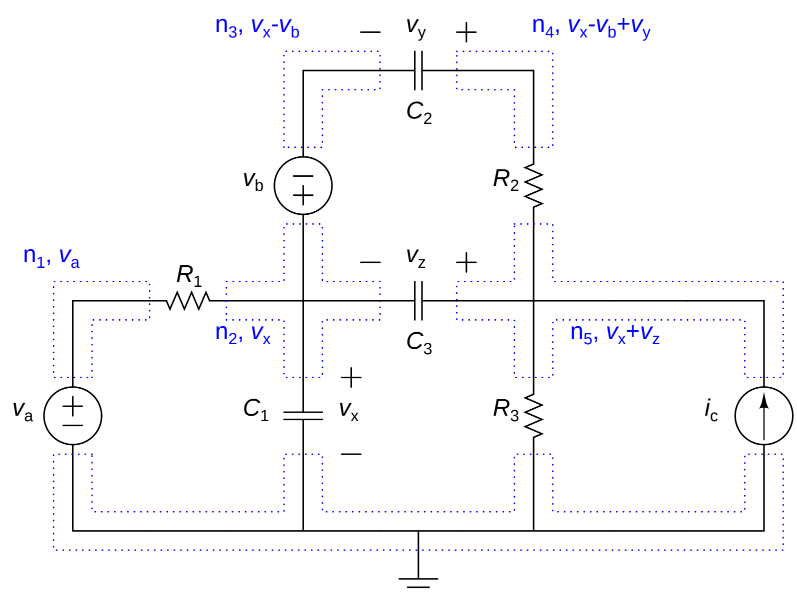
In circuits such as the one above, you may be given some initial set of voltages and currents for the sources and told at some time (say, t=0), one (or more) of the sources changes. The circuit will respond by having currents flow through the capacitors until the circuit has stabilized at its new values. The speed with which this happens depends upon the values of the capacitors and the resistors.

To solve the circuit in its entirety, then, you will need to get some set of initial conditions (as done above) as well as a set of differential equations to model the circuit. You can then solve the set of simultaneous differential equations in a manner very similar to that of solving simultaneous linear algebra equations - the only differences will be the presence of initial conditions in the solver function and the fact that the solver is called `dsolve` instead of `solve`.

To get the differential equations, first recall what the circuit looked like before the capacitors were removed:



then draw and label the nodes in terms of, first, known values and then labeled unknowns:



At this point there are three unknowns and six nodes to choose from. Unfortunately, only two of the nodes - nodes 4 and 5 - can be used since all other nodes touch voltage sources. This is where supernodes come in handy. The only supernodes you *cannot* consider are the ones that are dependent on nodes 4 and 5 - that eliminates supernode 45 and its supplement, supernode g123. Of the remaining two-node groupings, supernodes g2, g5, 12, 25, and 34 are invalid as they touch at least one voltage source. Supernodes g1 and 23 however are valid since they manage to avoid branches with voltage sources. Other two-node combinations are invalid since the nodes are not next to each other.

Now you can write the equations for the nodes. The equations for the single nodes are

$$\begin{aligned}
\text{KCL, n}_4: \quad & C_2\frac{d}{dt}\left(v_\mathrm{y}\right)+\frac{1}{R_2}\left(\left(v_\mathrm{x}-v_\mathrm{b}+v_\mathrm{y}\right)-\left(v_\mathrm{x}+v_\mathrm{z}\right)\right)=0\\
\text{KCL, n}_5: \quad & \frac{1}{R_2}\left(\left(v_\mathrm{x}+v_\mathrm{z}\right)-\left(v_\mathrm{x}-v_\mathrm{b}+v_\mathrm{y}\right)\right)+C_3\frac{d}{dt}\left(v_\mathrm{z}\right)+\frac{1}{R_3}\left(\left(v_\mathrm{x}+v_\mathrm{z}\right)-(0)\right)-i_\mathrm{c}=0
\end{aligned}$$

Next, pick the supernode equation. Supernode g1 cuts 4 branches (1 C, 2 R, 1 i) and supernode 23 cuts 4 branches (3 C, 1 R) so it is a matter of personal preference which to choose. With these six nodes, there are in fact a total of 25 "theoretical" supernodes containing between 2 and 5 nodes. For this problem, sticking with one of the 2-node versions will suffice. For completeness, they are *both* given below, though *only one* of them is required to solve the circuit - the Maple script only uses the supernode 23 equation, since *generally* the ground node should be avoided:

$$\begin{aligned}
\text{KCL, sn}_{g1}: \quad & C_1\frac{d}{dt}\left(-v_\mathrm{x}\right)+\frac{1}{R_1}\left(\left(v_\mathrm{a}\right)-\left(v_\mathrm{x}\right)\right)+\frac{1}{R_3}\left((0)-\left(v_\mathrm{x}+v_\mathrm{z}\right)\right)+i_\mathrm{c}=0\\
\text{KCL, sn}_{23}: \quad & C_1\frac{d}{dt}\left(v_\mathrm{x}\right)+\frac{1}{R_1}\left(\left(v_\mathrm{x}\right)-\left(v_\mathrm{a}\right)\right)+C_2\frac{d}{dt}\left(-v_\mathrm{y}\right)+C_3\frac{d}{dt}\left(-v_\mathrm{z}\right)=0
\end{aligned}$$

Again, choosing the supernode 23 equation, this leaves a set of three differential equations in terms of the known element values and sources and the three labeled unknowns:

$$\begin{aligned}
\text{KCL, n}_4: \quad & C_2\frac{d}{dt}\left(v_\mathrm{y}\right)+\frac{1}{R_2}\left(\left(v_\mathrm{x}-v_\mathrm{b}+v_\mathrm{y}\right)-\left(v_\mathrm{x}+v_\mathrm{z}\right)\right)=0\\
\text{KCL, n}_4: \quad & \frac{1}{R_2}\left(\left(v_\mathrm{x}+v_\mathrm{z}\right)-\left(v_\mathrm{x}-v_\mathrm{b}+v_\mathrm{y}\right)\right)+C_3\frac{d}{dt}\left(v_\mathrm{z}\right)+\frac{1}{R_3}\left(\left(v_\mathrm{x}+v_\mathrm{z}\right)-(0)\right)-i_\mathrm{c}=0\\
\text{KCL, sn}_{23}: \quad & C_1\frac{d}{dt}\left(v_\mathrm{x}\right)+\frac{1}{R_1}\left(\left(v_\mathrm{x}\right)-\left(v_\mathrm{a}\right)\right)+C_2\frac{d}{dt}\left(-v_\mathrm{y}\right)+C_3\frac{d}{dt}\left(-v_\mathrm{z}\right)=0
\end{aligned}$$

A tiny bit of tidying up algebraically - which is not explicitly required since Maple will take care of that for you - leaves:

$$\begin{aligned}
\text{KCL, n}_4: \quad & C_2\frac{d}{dt}\left(v_\mathrm{y}\right)+\frac{1}{R_2}\left(v_\mathrm{y}-v_\mathrm{b}-v_\mathrm{z}\right)=0\\
\text{KCL, n}_5: \quad & \frac{1}{R_2}\left(v_\mathrm{b}+v_\mathrm{z}-v_\mathrm{y}\right)+C_3\frac{d}{dt}\left(v_\mathrm{z}\right)+\frac{1}{R_3}\left(v_\mathrm{x}+v_\mathrm{z}\right)-i_\mathrm{c}=0\\
\text{KCL, sn}_{23}: \quad & C_1\frac{d}{dt}\left(v_\mathrm{x}\right)+\frac{1}{R_1}\left(v_\mathrm{x}-v_\mathrm{a}\right)-C_2\frac{d}{dt}\left(v_\mathrm{y}\right)-C_3\frac{d}{dt}\left(v_\mathrm{z}\right)=0
\end{aligned}$$

5.5 Solving Differential Equations and Plotting in Maple

See the Pundit page `Maple/Differential Equations` for more information on how to set up and solve ordinary differential equations in Maple. See the Pundit page `Maple/Plotting` for more information on how to make a plot in Maple.

5.6 Demonstration Code

There are several ways to put all this work together in a Maple script. There is a complete - albeit simple - example on Pundit at `Maple/Differential Equations/RC Example`. That page goes through the process of determining the DC steady state equations as well as the general model equations. It then shows a Maple script that can be used to solve for the initial conditions, to solve the differential equations, and to plot the results.

5.7 Assignment

There are two (new) parts to this assignment. For the first part, you will use Maple to solve for the unknown voltages in the sample circuit presented in Section 5.3 and then to make a single plot of all three capacitor voltages over time based on changes to the sources. For the second part, you will go through an *RC* circuit problem from scratch. Note - in all cases, you are assuming that the sources are at constant values before time 0. The values of the sources will change at time 0. For this assignment, they will change to different constants. Note however that these worksheets can be used if the sources are functions of time after $t = 0$ s. The assumption is that the circuit is in DC Steady State at time 0, so the sources do need to be constant for $t < 0$ s.

5.7.1 Sample Problem Completion

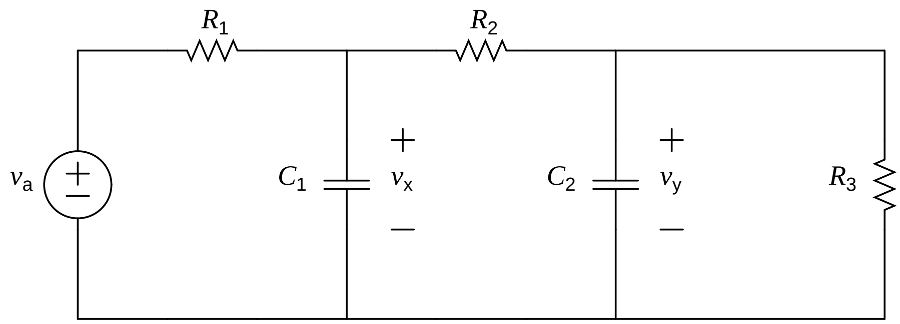
For this problem, you will be using the circuit introduced in Section 5.3, which resulted in the differential system given at the end of Section 5.4. Assume that $C_1 = 30$ μF, $C_2 = 30$ μF, $C_3 = 50$ μF, $R_1 = 10$ kΩ, $R_2 = 15$ kΩ and $R_3 = 10$ kΩ. Also assume that the sources (v_a, v_b, and i_c) had the following values:

Source	$t < 0$	$t \geq 0$
$v_a(t)$	5 V	0 V
$v_b(t)$	10 V	-8 V
$i_c(t)$	200 μA	900 μA

Complete a Maple worksheet that will solve the equations and then make two plots, each tracking all three capacitor voltages. The first graph should have a time span of 0 to 0.5 seconds − this will make it easier to check initial conditions. The second graph should have a time span of 0 to 10 seconds. Use three different linestyles and colors and be sure to have a legend in each plot. You should also include code to see what the final values for the capacitor voltages would be if the circuit were left in place for a long time. Think about what substitutions to make into already-solved equations that might simplify this process. Confirm that those values match the apparent asymptotes of your graph.

5.7.2 Same...Stuff, Different...Circuit

You will be repeating much of the process in the handout thus far, only on a different circuit. Specifically, you will be finding and plotting the answer for the capacitor voltages in the following circuit:



where $C_1 = 20\ \mu$F, $C_2 = 47\ \mu$F, $R_1 = 15$ kΩ, $R_2 = 10$ kΩ, and $R_3 = 5$ kΩ. The voltage source v_a is held at 10 V for a *really, really long time* before t=0, but at t=0 instantaneously changes to -5 V.

First, re-draw the circuit using DC equivalents and solve for the capacitor voltages in terms of the sources and the elements. Second, use KCL to find differential equations relating the capacitor voltages to the sources and possibly to each other. Then, complete a Maple worksheet that will plot the capacitor voltages as functions of time for 2 seconds. Use different linestyles and colors for each voltage and include a legend. You should also include code to see what the final values for the capacitor voltages would be if the circuit were left in place for a long time. Confirm that those values match the apparent asymptotes of your graph.

5.8 Lab Report

Your lab report for this week will consist of

(1) The MW file and a PDF of your Maple worksheet for completing the sample problem started in the lab handout. Your name, your lab section, the name of the assignment (Lab 5) and the name of the problem (Sample Problem Completion) should be in text at the top. You may also choose to annotate your worksheet, though that is not explicitly required. Be sure, however, that your plot is properly labeled.

(2) Your analysis of the Different...Circuit. Include your circuit drawings for finding both the DC steady-state and the coupled differential equations. This can be done by hand but should be neat and the picture you upload should be clear.

(3) The MW file and a PDF of your Maple worksheet for completing the Different...Circuit. Your name, your lab section, the name of the assignment (Lab 5) and the name of the problem (Different...Circuit) should be in text at the top. You may also choose to annotate your worksheet, though that is not explicitly required. Be sure, however, that your plot is properly labeled.