Laboratory 1: Solving Circuit Equations Using Maple

1.1 Introduction

This lab focuses on determining the linear algebra equations involved with solving electric circuits and on using Maple to find both the symbolic and the numeric solutions to those equations. Specifically, this lab depends on the full set of equations generated from Ohm's law and Kirchhoff's laws.

1.2 Resources

The additional resources required for this assignment include:

- Books: A&S
- Pratt Pundit Pages: `Maple/Initialization and Documentation`, `Maple/Simultaneous Equations`, `Maple/Examples/Circuits`,
  `EGR 224/Spring 2022/Lab 01`

1.3 Counting Equations

For an electric circuit containing only wires, sources, and resistances, there are three primary kinds of equations that will help solve for the individual voltages and currents that may exist in the circuit. These are: element equations, Kirchhoff's Current Law (KCL), and Kirchhoff's Voltage Law (KVL).

With respect to element equations, these can take multiple forms. For independent sources, the element equation is generally a value assigned to the source. For example, if there is a 10 V voltage source in the circuit, the element equation would be that the voltage across that element is 10 V. For resistances, the element equation is generally Ohm's Law

$$v = iR$$

so long as the voltage and current are labeled using the passive sign convention. Finally, for dependent sources, the element equation will relate a voltage or current in the circuit to the dependent variable of the source. There will be as many element equations as there are elements, though typically independent source values are counted as "knowns" rather than "element equations."

For Kirchhoff's Current Law, the number of independent equations that can be generated by applying it will be one fewer than the number of nodes in the circuit. Note that this does not explicitly mean that KCL must only be applied at simple nodes - "supernodes" which combine multiple nodes may also be used, though care must be taken to make sure the combination of equations constitute an independent set. Furthermore, there will be times when a node might be split into parts to determine currents flowing through specific wires that make up the node. For the brute force method, we will use simple nodes.

For Kirchhoff's Voltage Law, the number of independent equations that can be derived in a circuit consisting solely of two-terminal elements will be equal to:[1]

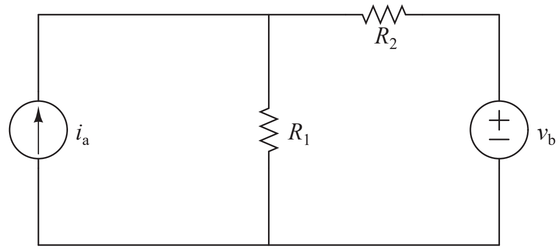
$$\text{KVL equations} = \text{elements} - \text{nodes} + 1$$

For planar circuits, this will be equal to the number of meshes in the circuit. There may be times when it is more efficient to use some superloop equations versus mesh equations, but for the brute force method with a planar circuit, we will use mesh equations.

[1] Peter Feldmann and Ronald A. Rohrer, "Proof of the Number of Independent Kirchhoff Equations in an Electrical Circuit." http://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=00135739

1.4 Example

The following example will demonstrate how to determine the required *number* of equations for the brute force method as well as what those equations might be. Consider the following circuit:



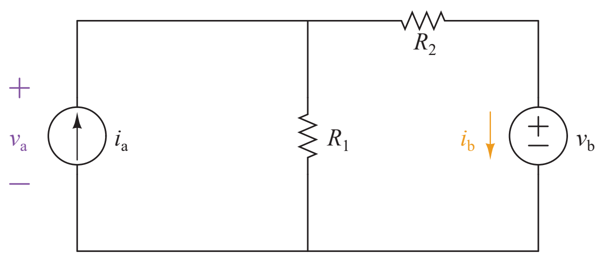
Given that there are four elements, there are a total of eight potential unknowns - four element voltages and four element currents. There are four elements and three nodes, meaning the number and type of equations we can use is:

- 4 elements → 4 element equations
- 3 nodes - 1 → 2 independent KCL equations
- 4 elements - 3 nodes + 1 → 2 independent KVL equations (also, this circuit is planar, so 2 meshes → 2 independent KVL equations)

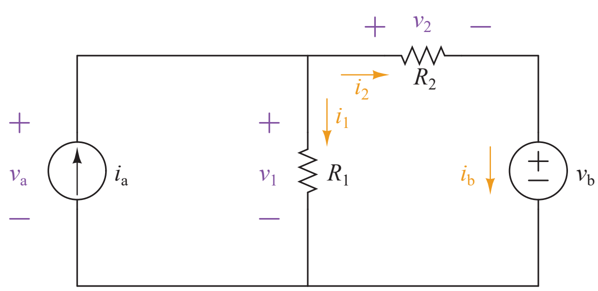
Given the two independent sources, it is likely that the "element equations" for those two items will instead be replaced with known values for the current source's current and the voltage source's voltage. That leaves six equations and six unknowns.

1.4.1 Variable Labels

Before writing equations, label the known and unknown values. For independent sources, you may choose either passive or active sign convention - just be sure to remember which is which. In the example below, element i_a is labeled with the active sign convention while element v_b is labeled with the passive sign convention.



For the resistive elements, either the direction of the current measurement or the direction of the voltage drop measurement can be labeled first; once one variable is labeled, however, the other needs to follow the passive sign convention. For this example, the remaining four items are labeled as:



1.4.2 Element Equations

Assuming the independent source values are known, there are two element equations:

$$\text{Ohm's Law, } R_1: \; v_1 = i_1 R_1$$
$$\text{Ohm's Law, } R_2: \; v_2 = i_2 R_2$$

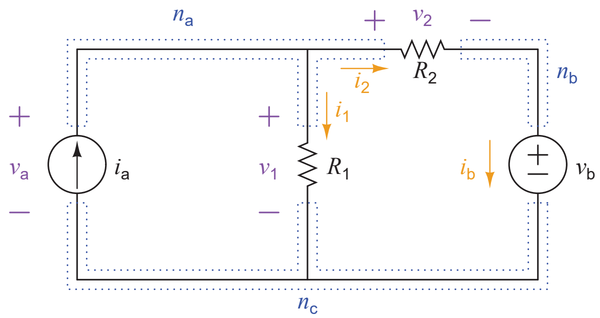
If we were to want to solve for all the circuit variables strictly using linear algebra equations set up in matrix form, these equations would become:

$$v_1 - i_1 R_1 = 0$$
$$v_2 - i_2 R_2 = 0$$

to make sure that all the unknown variables are on the left. Maple, however, does not have such a requirement, so either form will work.

1.4.3 KCL Equations

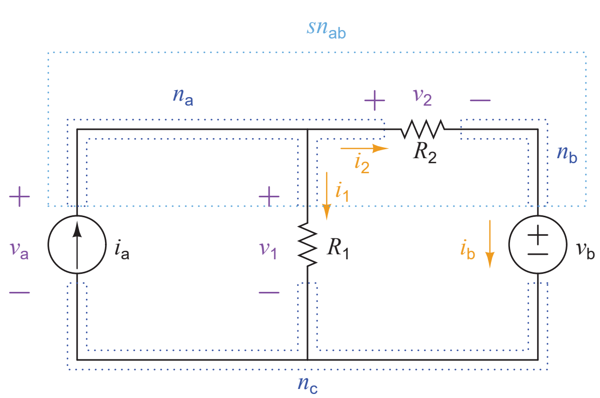
With 3 nodes, there are two *independent* KCL equations. As noted earlier, however, there may be more than two ways to write them. For this circuit, consider the following three nodes:



The KCL equations at each node, determined by adding up the total current *leaving* the node, would be:

$$\begin{aligned} \text{KCL, } n_\text{a}: &\quad -i_\text{a} + i_1 + i_2 = 0 \\ \text{KCL, } n_\text{b}: &\quad -i_2 + i_\text{b} = 0 \\ \text{KCL, } n_\text{c}: &\quad i_\text{a} - i_1 - i_\text{b} = 0 \end{aligned}$$

Note that these equations are not independent. Specifically adding all three together yields 0, a sure sign of dependence. Another way to look at this is to consider the current leaving the "supernode" combination of n_a and n_b, labeled as sn_ab below:



The KCL equation for this gives:

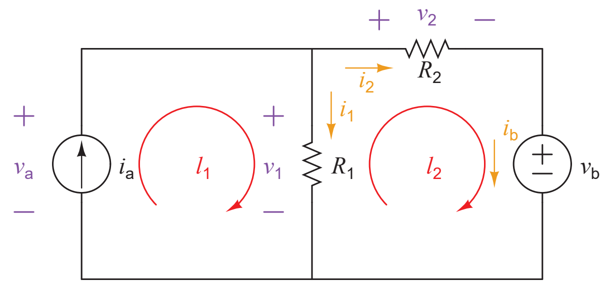
$$\text{KCL, } sn_\text{ab}: \quad -i_\text{a} + i_1 + i_\text{b} = 0$$

Essentially, i_2 is taken out of consideration because it is internal to the supernode. That which remains is equal and opposite to the current leaving n_c and is therefore not independent of it.

For more complex circuits, there may be a large number of supernodes to consider - sometimes, a supernode can give a much more useful equation than the nodes interior to it. Just make sure the equations used are all independent of each other. For this example, only the n_a and n_b equations will carry over to the final analysis.

1.4.4 KVL Equations

Since this is a planar circuit with two meshes, there are 2 independent KVL equations. There are, however, three different loops - one for each mesh and another "superloop" made by combining the meshes together. That is, given



with meshes labeled l_1 and l_2, the three loop equations (counting voltage drops) would be:

$$\begin{aligned} \text{KVL, } l_1: &\quad -v_\text{a} + v_1 = 0 \\ \text{KVL, } l_2: &\quad -v_1 + v_2 + v_\text{b} = 0 \\ \text{KVL, } sl_{12}: &\quad -v_\text{a} + v_2 + v_\text{b} = 0 \end{aligned}$$

In this case, note that the sum of the two mesh equations is equal to the superloop equation - another sign of dependence. There will be times that a superloop equation produces a more useful or simpler equation than those of its component meshes. This, however, is not one of those times so only two two mesh equations will show up in the final list.

1.4.5 Summary of Equations

At this stage, there are a total of six unknowns: v_a, v_1, v_2, i_b, i_1, and i_2. This requires six independent equations. From the three sections above, we have:

$$\begin{aligned} \text{Ohm's Law, } R_1: &\quad v_1 = i_1 R_1 \\ \text{Ohm's Law, } R_2: &\quad v_2 = i_2 R_2 \\ \text{KCL, } n_\text{a}: &\quad -i_\text{a} + i_1 + i_2 = 0 \\ \text{KCL, } n_\text{b}: &\quad -i_2 + i_\text{b} = 0 \\ \text{KVL, } l_1: &\quad -v_\text{a} + v_1 = 0 \\ \text{KVL, } l_2: &\quad -v_1 + v_2 + v_\text{b} = 0 \end{aligned}$$

1.5 Solving Equations

Once you have found the necessary equations, there are multiple methods for solving them. If you choose to use matrix methods, you will need to organize the equations such that all the unknowns (and their coefficients) are on one side and the known items are on the other. You can then set up and solve a matrix equation using whatever method you like (Cramer's Rule, Gaussian Elimination, etc.). For the six equations above, this would yield:

$$\begin{aligned} v_1 - i_1 R_1 &= 0 \\ v_2 - i_2 R_2 &= 0 \\ i_1 + i_2 &= i_\text{a} \\ i_\text{b} - i_2 &= 0 \\ v_\text{a} - v_1 &= 0 \\ v_1 - v_2 &= v_\text{b} \end{aligned}$$

which, in matrix form, becomes:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & -R_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -R_2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_\text{a} \\ v_1 \\ v_2 \\ i_\text{b} \\ i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ i_\text{a} \\ 0 \\ 0 \\ v_\text{b} \end{bmatrix}$$

If you have access to a computer, however, you may consider using a computational tool to do the algebra for you. One such tool is Maple, produced by Maplesoft.

1.6 Introduction to Maple

There are several Pratt Pundit pages that are (hopefully!) useful for learning / relearning the basics of Maple. You should go through these before starting to work on the specific tasks for this lab. We will also cover them during the lab period. They are:

- `Maple/Initialization and Documentation`
  This page goes through starting Maple, clearing the memory, and documenting your work.
- `Maple/Simultaneous Equations`
  This page goes through setting up and solving simultaneous equations, substituting in values, and simplifying results.

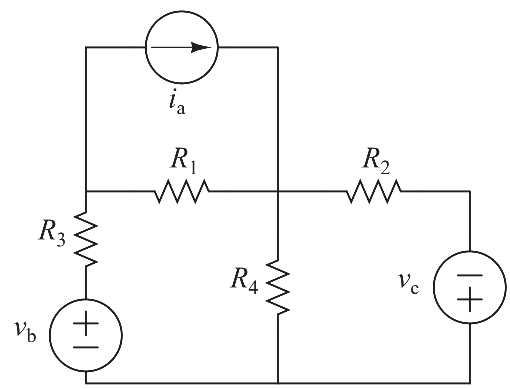
In addition, a Maple worksheet that solves for the voltages, currents, and powers for the elements in the example circuit is at `Maple/Examples/Circuits`. Finally, there are some examples on the Sakai page in the Resources section.

1.7 Assignment

You are going to use Maple and the brute force method to solve for the currents through and voltages across each element in two different circuits.

1.7.1 Based on Problem 3.14 (Figure 3.63) in the book

Start with the following symbolic representation, assuming the independent source values are known:



(1) Label all unknown elemental voltage drops and currents using passive sign convention. Instead of calling the voltage across the 4 Ω resistor v_o, as in the book, use v_{R_4}

(2) Circle and label all five nodes.

(3) Name all three meshes.

(4) Generate a list of all four element equations.

(5) Generate a list of all five KCL equations. State which one you are going to throw out.

(6) Generate a list of all three mesh-based KVL equations.

(7) Using Maple:

(a) Symbolically solve for all eleven unknown variables. Use `simplify` and `expand` on your list of answers to get the simplest representation. Call this set of solutions `MySoln`. Check to make sure the units make sense with these solutions.

(b) Symbolically calculate equations for the power delivered by each of the three sources and put this equations in a list called `AuxEqn`. These should be *very simple* representations using your known and unknown variables - do *not* substitute in your symbolic solutions from `MySoln` yet! For example, the list with the equation for the power delivered by v_a filled in looks like:

```
AuxEqn:=[pdelva=-va*ia, pdelvb=???, pdelvc=???]
```

even though v_a is an unknown.

(c) Calculate the numerical values of the voltages and currents. The easiest way to do this is to create a variable called `Vals` that will store both the passive circuit element values and the known source values. Use the `subs` command to substitute all those values into your symbolic solution list for the voltages and currents. Display your final answer with 4 significant figures (i.e. use the `evalf[4]` command).

(d) Calculate the numerical values for the powers described above. Display your final answer with 4 significant figures (i.e. use the `evalf[4]` command). Here, you will want to first substitute the symbolic solutions for your voltages and currents into your auxiliary equations, then you will substitute the numerical values from `Vals` into that.

In your Maple worksheet, be sure to include your name, lab section, and the assignment name at the top. You should also clearly indicate where in your worksheet the answer for each section is by putting a comment just above it; for example,

```
Answer to part (7a): symbolic solutions for unknown variables
```

When you turn in your lab, be sure you include a picture of your labeled circuit drawing as well as your Maple worksheet in both MW and PDF forms. Also make sure your variables are clearly shown on your drawing and that they match those used in the worksheet. For parts (4)-(6) above, the code you use to define the element, KCL, and KVL equations will be sufficient for showing that you have generated them.

1.7.2 Based on Problem 3.31 (Figure 3.80) in the book

Perform the same tasks for this circuit as you did for the one above. Before beginning, create a symbolic representation of the circuit that you can use to come up with the symbolic equations, then follow the same seven steps as for the first circuit. Note that two of the unknowns already have names - v_{o} is the voltage across the horizontal 1 Ω resistor and I_{o} is the current through the right 4 Ω resistor. If you choose to use different labels for these, be sure to also change the labels in the dependent sources. For example, I would most likely call the top horizontal resistor R_1 and this the voltage across it would be v_1. Based on that, I would change the formula for the voltage controlled current source to $i = g\ v_1$ in the symbolic version of the circuit. Ignore the node voltage labels given for the problem.

This circuit has nine elements, meaning there are eighteen electrical quantities to keep track of. There are a total fourteen unknowns - five resistor voltages, five resistor currents, the currents through each of the two voltage sources, and the voltages across each of the two current sources. There are therefore four known quantities. It turns out that your controlling variables are already accounted for as elemental values (that is, the controlling voltage for the voltage controlled current source happens to also be a resistor voltage, and the controlling current for the current controlled voltage source happens to also be a resistor current). There are also two known independent source values.

Somewhere in the middle of your worksheet, you should have the numerical values for the fourteen (previously unknown) voltages and currents, rounded to four significant figures. You do not need to *show* these results symbolically but I recommend leaving the colon off at first to make sure you get solutions. Do a quick units check, then add a colon to the end of that line if you are satisfied that the symbolic answers are reasonable.

At the end of your worksheet, you should have numerical values for the powers delivered by each of the four sources (two independent and two dependent), rounded to four significant figures. You do not need to show these symbolically. Be sure to turn in both a PDF and MW version of your Maple code as well as your symbolic circuit drawing.