1 - DRAFT SOLUTIONS for TEST II (Fall 2024)

$$w(t) = \cos(10t) + \sin(8t)\cdot\cos(14t)$$

$$w(t) = \cos(10t) + \frac{1}{2}\left(\sin(-6t) + \sin(22t)\right)$$

$$w(t) = \cos(10t) - \frac{1}{2}\sin(6t) + \frac{1}{2}\sin(22t)$$

$\omega = 10$ $\omega = 6$ $\omega = 22$ $\omega_0 = 2$

$$W[k] = \begin{cases} k = 11 & 1/(4j) \text{ or } -j/4 \\ 5 & 1/2 \\ 3 & -1/(4j) \text{ or } j/4 \\ -3 & 1/(4j) \text{ or } -j/4 \\ -5 & 1/2 \\ -11 & -1/(4j) \text{ or } j/4 \end{cases}$$

2) $T = 4$, $\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

$$X[k] \begin{cases} k = 7 & -6 \\ k = 2 & -j5 = 5/j \\ k = 0 & 4 \\ k = -2 & j5 = -5/j \\ k = -7 & -6 \end{cases}$$

$$-12\cos\left(\frac{7\pi}{2}t\right) + 10\sin(\pi t) + 4$$

2

$$y(t) = \sum_{n=-\infty}^{\infty} r(t-5n) - 2u(t-5n-1) - r(t-5n-3) - u(t-5n-4)$$

$T = 5$

$w_0 = \frac{2\pi}{5}$

$$Y[0] = \frac{(\frac{1}{2}) + (-\frac{1}{2}) + (\frac{1}{2}) + (1)}{5} = \frac{3}{10}$$

$k \neq 0$:

$$Y[k] = \frac{1}{(jkw_0)^2 T} - \frac{2e^{-jkw_0}}{(jkw_0)T} - \frac{e^{-j3kw_0}}{(jkw_0)^2 T} - \frac{e^{-j4kw_0}}{(jkw_0)T}$$

or

$$\frac{-T}{4\pi^2 k^2} - \frac{2e^{-jkw_0}}{2\pi jk} + \frac{Te^{-j3kw_0}}{4\pi^2 k^2} - \frac{e^{-j4kw_0}}{2\pi jk}$$

3 (a, b)

(a) $(e^{-t} - t)\cdot(u(t)-u(t-2))$

$e^{-t}u(t) - t\,u(t) - e^{-t}u(t-2) + t\,u(t-2)$

$e^{-t}u(t) - t\,u(t) - e^{-((t-2)+2)}u(t-2) + ((t-2)+2)u(t-2)$

$e^{-t}u(t) - t\,u(t) - e^{-2}e^{-(t-2)}u(t-2) + (t-2)u(t-2) + 2u(t-2)$

$$A(j\omega) = \frac{1}{j\omega+1} - \frac{1}{(j\omega)^2} - \frac{e^{-2}e^{-j2\omega}}{j\omega+1} + \frac{e^{-j2\omega}}{(j\omega)^2} + \frac{2e^{-j2\omega}}{j\omega}$$

ALT: $a_1(t)$ $u(t)-u(t-2)$ General rectangular pulse, $a=0, b=2$ or $\text{rect}(\frac{t-1}{2}) = \text{rect}(\frac{t}{2}) * \delta(t-1)$

$A_1(j\omega) = \frac{2\sin(\omega)}{\omega}(e^{-j\omega}) = \frac{2}{\omega}\left(\frac{e^{j\omega}-e^{-j\omega}}{2j}\right)e^{-j\omega} = \frac{1}{j\omega}(1-e^{-2j\omega})$

$a_2(t) = e^{-t}a_1(t)$ $\quad A_2(j\omega) = A_1(j(\omega-j)) = \frac{1}{j(\omega-j)}(1-e^{-2j(\omega-j)}) = \frac{1}{j\omega+1}(1-e^{-2j\omega}e^{-2})$

$j\omega_0=-1,\ \omega_0 = j$

$a_3(t) = -t\,a_1(t)$ $\quad A_3(j\omega) = -j\frac{d}{d\omega}A_1(j\omega) =$

$-j\left(\left(\frac{-1}{j\omega^2}(1-e^{-2j\omega})\right) + \frac{1}{j\omega}(2je^{-2j\omega})\right)$

$-j = \frac{1}{j}$, so $\quad \frac{-1}{(j\omega)^2} + \frac{e^{-2j\omega}}{(j\omega)^2} + \frac{2e^{-2j\omega}}{j\omega}$

(b) $b(t) = \cos(4t)\cdot\sin(3t)$

$B(j\omega) = \frac{1}{2\pi}\left(\pi(\delta(\omega-4)+\delta(\omega+4))\right) * \left(\frac{\pi}{j}(\delta(\omega-3)-\delta(\omega+3))\right)$

$= \frac{\pi}{2j}\left(\delta(\omega-7) - \delta(\omega-1) + \delta(\omega+1) - \delta(\omega+7)\right)$

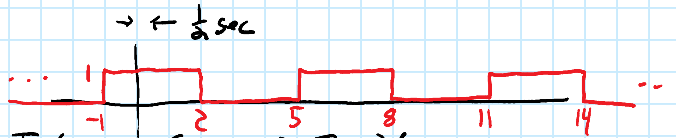
Alt: $b(t) = \frac{1}{2}\left(\sin(-t) + \sin(7t)\right)$

$= \frac{1}{2}\left(-\sin(t) + \sin(7t)\right)$

$B(j\omega) = \frac{1}{2}\left(-\frac{\pi}{j}(\delta(\omega-1) + \delta(\omega+1)) + \frac{\pi}{j}(\delta(\omega-7)+\delta(\omega+7))\right)$

3 (c)

$$c(t) = \sum_{n=-\infty}^{\infty} u(t+1-6n) - u(t-2-6n)$$



$T=6,\ \omega_0 = \pi/3,\ W=3,\ T_1 = 3/2$

1) PERIODIC RECTANGLE WAVE (WIDTH=3) SHIFTED RIGHT 1/2 SECOND

$$C(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\sin\left(k\cdot\frac{\pi}{3}\cdot\frac{3}{2}\right)}{k} \delta\left(\omega-k\tfrac{\pi}{3}\right) e^{-jk\frac{\pi}{3}\cdot\frac{1}{2}} = \sum_{k=-\infty}^{\infty} \frac{2\sin\left(k\frac{\pi}{2}\right)\delta\left(\omega-k\frac{\pi}{3}\right)e^{-jk\frac{\pi}{6}}}{k}$$

GIVEN $\delta(\omega-k\frac{\pi}{3})$ ALL k OUT OF $\delta(\,)$ CAN BE REPLACED WITH $3\omega/\pi$:

$$C(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\sin\left(\omega\cdot\frac{3}{2}\right)\delta\left(\omega-k\frac{\pi}{3}\right)e^{-j\omega\cdot\frac{1}{2}}}{(3\omega/\pi)} = \frac{2\pi}{3}\,\frac{\sin\left(\omega\cdot\frac{3}{2}\right)e^{-j\omega\cdot\frac{1}{2}}}{\omega} \sum_{k=-\infty}^{\infty} \delta\left(\omega-k\tfrac{\pi}{3}\right)$$

2) RECTANGULAR PULSES, SHIFTED WITH IMPULSE TRAIN

$$c(t) = \sum_{n=-\infty}^{\infty} \hat{c}(t-6n) \qquad \hat{c}(t) = \underset{a=-1}{u(t+1)} - \underset{b=2}{u(t-2)} \qquad \hat{C}(j\omega) = \frac{2\sin\left(\omega\cdot\frac{3}{2}\right)}{\omega} e^{-j\omega\frac{1}{2}}$$

$$c(t) = \hat{c}(t) * \sum_{n=-\infty}^{\infty}\delta(t-6n) \qquad C(j\omega) = \hat{C}(j\omega)\cdot \frac{2\pi}{6}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{6}\right) = \frac{2\sin\left(\omega\cdot\frac{3}{2}\right)e^{-j\omega\frac{1}{2}}}{\omega}\cdot\frac{\pi}{3}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{k\pi}{3}\right)$$

$$C(j\omega) = \frac{2\pi}{3}\,\frac{\sin\left(\omega\cdot\frac{3}{2}\right)e^{-j\omega\frac{1}{2}}}{\omega}\sum_{k=-\infty}^{\infty}\delta(\omega-k\pi/3)$$

3) SINGULARITIES:

$$c(t) = \sum_{n=-\infty}^{\infty} u(t+1-6n) - u(t-2-6n)$$

$$C(j\omega) = \left(\sum_{n=-\infty}^{\infty}\left(\frac{1}{j\omega}\right)e^{j\omega(1-6n)} - \left(\frac{1}{j\omega}\right)e^{j\omega(-2-6n)}\right)$$

$$= \sum_{n=-\infty}^{\infty} \frac{e^{-j\omega\cdot\frac{1}{2}}e^{-j\omega\cdot 6n}}{j\omega}\left(e^{j\omega\cdot\frac{3}{2}} - e^{-j\omega\cdot\frac{3}{2}}\right)$$

$$= \sum_{n=-\infty}^{\infty} \frac{e^{j\omega\left(\frac{1}{2}-6n\right)}}{j\omega}\,2j\sin\left(\omega\cdot\tfrac{3}{2}\right)$$

$$= \frac{2\sin\left(\omega\cdot\frac{3}{2}\right)e^{j\omega\frac{1}{2}}}{\omega}\sum_{n=-\infty}^{\infty}e^{-j\omega 6n}$$

COOL MATH TRICKS

$$\sum_{n=-\infty}^{\infty}\delta(t-nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty}e^{jkt\cdot\frac{2\pi}{T}}$$

$$\sum_{n=-\infty}^{\infty}\delta\left(t-n\tfrac{2\pi}{\omega_0}\right) = \frac{\omega_0}{2\pi}\sum_{k=-\infty}^{\infty}e^{jk\omega_0 t}$$

$$\sum_{n=-\infty}^{\infty}e^{jnt_0\omega} = \frac{2\pi}{t_0}\sum_{k=-\infty}^{\infty}\delta\left(\omega-k\tfrac{2\pi}{t_0}\right)$$

SINCE n, k GO $[-\infty, \infty]$, EITHER/BOTH CAN HAVE SIGN FLIPPED.

FROM FOURIER SERIES $\displaystyle\sum_{n=-\infty}^{\infty}\delta(t-nT) = \sum_{k=-\infty}^{\infty}\frac{1}{T}e^{jk\omega_0 t}$ $\qquad T=\frac{2\pi}{\omega_0}$

TRANSFORM VARIABLES
$k \leftrightarrow n$ $T\to\Omega$ $t\to\omega$
$\omega_0\to t_0$

$$\sum_{k=-\infty}^{\infty}\delta(\omega-k\Omega) = \sum_{n=-\infty}^{\infty}\frac{1}{\Omega}e^{jnt_0\omega} \qquad t_0=6 \quad \Omega=\frac{2\pi}{t_0}=\frac{\pi}{3}$$

REVERSE ORDER
MULTIPLY BY Ω

$$\sum_{n=-\infty}^{\infty}e^{jnt_0\omega} = \Omega\sum_{k=-\infty}^{\infty}\delta(\omega-k\Omega) = \frac{\pi}{3}\sum_{k=-\infty}^{\infty}\delta\left(\omega-k\tfrac{\pi}{3}\right)$$

$$= \frac{2\sin\left(\omega\cdot\frac{3}{2}\right)e^{-j\omega\frac{1}{2}}}{\omega}\,\frac{\pi}{3}\sum_{k=-\infty}^{\infty}\delta\left(\omega-k\tfrac{\pi}{3}\right)$$

4

$$W(j\omega) = \frac{5e^{-j3\omega}}{j\omega+4} \qquad w(t) = 5e^{-4(t-3)}u(t-3)$$

$$X(j\omega) = \frac{4j\omega+30}{(j\omega)^2+6j\omega+34} = \frac{4j\omega+30}{(j\omega+3)^2+(5)^2}$$

$b^2-4ac = 36-136<0$: $\underline{\text{MOAT}}$

$$(j\omega+3)^2 = (j\omega)^2+6j\omega+\underbrace{9+25}_{34}$$

$$A(j\omega+3)+B(5) = 4j\omega+30$$

$$A=4 \qquad \cancel{4j\omega}+12+5B = \cancel{4j\omega}+30$$

$$B = \frac{30-12}{5} = \frac{18}{5} = 3.6$$

$$x(t) = e^{-3t}(4\cos 5t + 3.6\sin 5t)u(t)$$

$$Y(j\omega) = \frac{(j\omega)^2}{(j\omega)^2+7j\omega+12} = \frac{(j\omega)^2+7j\omega+12-7j\omega-12}{(j\omega)^2+7j\omega+12}$$

$$= 1+\frac{-7j\omega-12}{(j\omega)^2+7j\omega+12} = 1+\frac{-7j\omega-12}{(j\omega+3)(j\omega+4)} = 1+\frac{A}{j\omega+3}+\frac{B}{j\omega+4}$$

$b^2-4ac = 49-48>0$: FACTOR

$$A = \lim_{j\omega\to-3}\frac{-7j\omega-12}{j\omega+4} = \frac{21-12}{1} = 9$$

$$B = \lim_{j\omega\to-4}\frac{-7j\omega-12}{j\omega+3} = \frac{28-12}{-1} = -16$$

$$Y(j\omega) = 1+\frac{9}{j\omega+3}-\frac{16}{j\omega+4}$$

$$y(t) = \delta(t)+9e^{-3t}u(t)-16^{-4t}u(t)$$

ALT: $Y(j\omega) = j\omega\left(\frac{j\omega}{(j\omega)^2+7j\omega+12}\right) = j\omega\left(\frac{j\omega}{(j\omega+3)(j\omega+4)}\right)$

$= j\omega\left(\frac{A}{j\omega+3}+\frac{B}{j\omega+4}\right) \quad A = \lim_{j\omega\to-3}\frac{j\omega}{j\omega+4} = -3 \quad B = \lim_{j\omega\to-4}\frac{j\omega}{j\omega+3} = 4$

$= j\omega\left(\frac{-3}{j\omega+3}+\frac{4}{j\omega+4}\right)$

1 if t=0   1 if t=0

$$y(t) = \frac{d}{dt}\left(-3e^{-3t}u(t)+4e^{-4t}u(t)\right) = 9e^{-3t}u(t)\cancel{-3e^{-3t}}\delta(t)-16e^{-4t}u(t)+\cancel{4e^{-4t}}\delta(t)$$

$$= \delta(t)+9e^{-3t}u(t)-16e^{-4t}u(t)$$

ALT 2: $Y(j\omega) = (j\omega)^2\left(\frac{1}{(j\omega)^2+7(j\omega)+12}\right) = (j\omega)^2\left(\frac{1}{(j\omega+3)(j\omega+4)}\right) = (j\omega)^2\left(\frac{1}{j\omega+3}+\frac{-1}{j\omega+4}\right)$

1 if t=0   1 if t=0

$$y(t) = \frac{d}{dt}\left(\frac{d}{dt}\left(e^{-3t}u(t)-e^{-4t}u(t)\right)\right) = \frac{d}{dt}\left(-3e^{-3t}u(t)+\cancel{e^{-3t}}\delta(t)+4e^{-4t}u(t)-\cancel{e^{-4t}}\delta(t)\right)$$

$$= \delta(t)+9e^{-3t}u(t)-16e^{-4t}u(t)$$

$$3y[n] = 6x[n] - 9x[n-2] - y[n-1]$$

$$y[n] + \frac{1}{3}y[n-1] = 2x[n] - 3x[n-2]$$

$$h_0[n] + \frac{1}{3}h_0[n-1] = \delta[n]$$

$$K\gamma^0 + \frac{1}{3}K\gamma^{-1} = 0 \quad \gamma = -1/3$$

$$n=0 \quad h_0[0] + \frac{1}{3}h[-1] = \delta[0] = 1$$

$$K + 0 = 1 \quad K = 1$$

$$h_0[n] = K\gamma^n u[n] = \left(-\frac{1}{3}\right)^n u[n]$$

$$h[n] = 2h_0[n] - 3h_0[n-2]$$

$$= 2\left(-\frac{1}{3}\right)^n u[n] - 3\left(-\frac{1}{3}\right)^{n-2} u[n-2]$$

2. $s_r[n] = h[n] * u[n]$

$$\alpha^n u[n] * u[n] = \left(\frac{\alpha^{n+1} - 1}{\alpha - 1}\right) u[n]$$

$$\text{or } \left(\frac{1 - \alpha^{n+1}}{1 - \alpha}\right) u[n]$$

$$\left(2\left(-\frac{1}{3}\right)^n u[n] - 3\left(-\frac{1}{3}\right)^{n-2} u[n-2]\right) * u[n]$$

$$2\left(\frac{1 - \left(-\frac{1}{3}\right)^{n+1}}{1 + \frac{1}{3}}\right) u[n] - 3\left(\frac{1 - \left(-\frac{1}{3}\right)^{n-1}}{1 + \frac{1}{3}}\right) u[n-2]$$

$$\frac{3}{2}\left(1 - \left(-\frac{1}{3}\right)^{n+1}\right) u[n] - \frac{9}{4}\left(1 - \left(-\frac{1}{3}\right)^{n-1}\right) u[n-2]$$

6

1) $$3\frac{dy}{dt} + 6y = 4x$$

$$3j\omega Y + 6Y = 4X$$

$$\frac{Y}{X} = \frac{4}{3j\omega + 6}$$

2) $$3\frac{ds_r(t)}{dt} + 6s_r(t) = 4u(t) \qquad s_r(0) = 0$$

$$\frac{1}{2}\frac{ds_r(t)}{dt} + s_r(t) = \frac{2}{3} \qquad \tau = \frac{1}{2} \quad s_r(\infty) = \frac{2}{3}$$

$$s_r(t) = s_r(\infty) + (s_r(0) - s_r(\infty))\, e^{-t/\tau} u(t)$$

$$s_r(t) = \frac{2}{3}(1 - e^{-2t}) \quad u(t)$$

3) $$h(t) = \frac{ds_r(t)}{dt} = \underbrace{\frac{2}{3}(1 - e^{-2t})}_{\to 0}\delta(t) + \frac{2}{3}(2e^{-2t})u(t)$$

$$= \boxed{\frac{4}{3}e^{-2t}u(t)}$$

ALT: $$H(j\omega) = \frac{4}{3j\omega + 6} = \frac{4/3}{j\omega + 2} \qquad h(t) = \mathcal{F}^{-1}\{H(j\omega)\} = \boxed{\frac{4}{3}e^{-2t}u(t)}$$

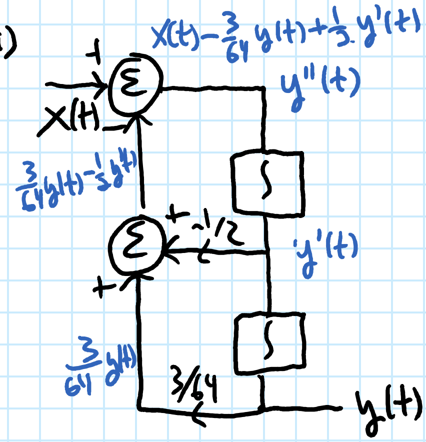
$$S_r(j\omega) = \frac{H}{j\omega} = \frac{4/3}{(j\omega)(j\omega+2)} = \frac{A}{j\omega} + \frac{B}{j\omega + 2}$$

$$A = \lim_{j\omega \to 0} \frac{4/3}{j\omega + 2} = \frac{2}{3} \qquad B = \lim_{j\omega \to -2} \frac{4/3}{j\omega} = -\frac{2}{3}$$

$$s_r(t) = \boxed{\frac{2}{3}u(t) - \frac{2}{3}e^{-2t}u(t)}$$

7

(a)



$$y''(t) = x(t) + \frac{1}{2}y'(t) - \frac{3}{64}y(t)$$

$$y''(t) - \frac{1}{2}y'(t) + \frac{3}{64}y(t) = x(t)$$

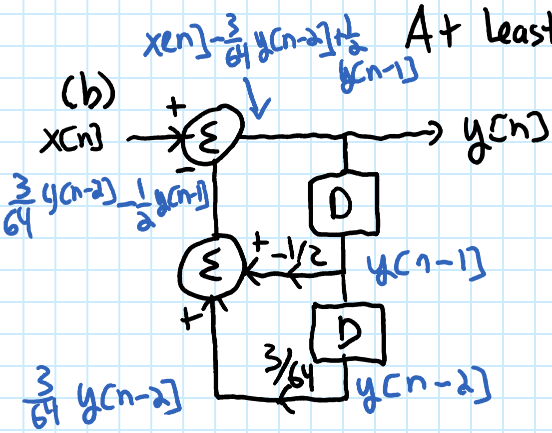
$$s^2 - \frac{1}{2}s + \frac{3}{64} = 0$$

$$s = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{12}{64}}}{2}$$

$$= \frac{\frac{1}{2} \pm \sqrt{\frac{4}{64}}}{2} = \frac{1}{4} \pm \frac{1}{8}$$

At least one real part of $s > 0$; unstable

(b)



$$y[n] = x[n] + \frac{1}{2}y[n-1] - \frac{3}{64}y[n-2]$$

$$y[n] - \frac{1}{2}y[n-1] + \frac{3}{64}y[n-2] = x[n]$$

$$1 - \frac{1}{2}z^{-1} + \frac{3}{64}z^{-2} = 0$$

$$z^2 - \frac{1}{2}z + \frac{3}{64} = 0$$

$$z = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{12}{64}}}{2} = \frac{1}{4} \pm \frac{1}{8}$$

both $|z| < 1$ BIBO STABLE